CALCULUS OF FINITE DIFFERENCES

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OF

FINITE DIFFERENCES

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PRAMATHA NATH MITRA, M.A., B.L.

LECTURER IN PURE MATHEMATICS IN THE UNIVERSITY OF CALCUTTA

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PREFACE

The present volume has evolved out of the lectures delivered by me to the Post-Graduate students of Pure Mathematics in the University of Calcutta. It aims at giving a fair, comprehensive and connected account of the subject of Finite Differences.

The subject-matter has been presented in a form which will be easily understood, and the different problems have been demonstrated in as simple a manner as the topic under discussion would allow. In many places new and original proofs have been given. The needs for beginners in the subject have been kept in view so that the book will be found suitable for beginners as also for advanced students and researchers.

A comprehensive account of symbolic operators, of finite and approximate summations, and all the various propositions of finite differences have been given. Interpolation has been discussed at greater length. Difference equation has not been treated in this volume, as the same is intended to find place in a separate volume.

Every branch of scientific enquiry has got a practical side but unfortunately in most cases little or no stress is given in that direction so that the learner knows the theory but cannot make practical applications of his knowledge. The theories of Finite Differences admit of wider applications. They are required by scholars in many different fields—Astronomy, Actuary, Engineering, Statistics, Physics, Meteorology.......as well as Pure Mathematics. An attempt has been made here to combine theory with practice by bringing the ignored practical side into some prominence. The subject-matter covers an extensive field and has been amply illustrated by numerical examples and applied to data taken from observations—actuarial, astronomical......so that the worker may learn how to apply his theories to diverse problems.

The present volume has been designed to meet the requirements of students of Pure Mathematics as also of students preparing for actuarial, statistical and other natural science examinations.

In the preparation of this work I had to consult the existing treatises and various memoirs and research papers on the subject and my thanks are due to their respective authors. My own researches also have been incorporated in proper places in the body of the book.

Any suggestion for improvement will be thankfully received.

In conclusion, I wish to express my thanks to the officers and the staff of the Calcutta University Press for the pains they have taken in printing off the book.

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P. N. MITRA.

CHAPTER I

DIFFERENCE OPERATORS

1.1. Introduction. Calculus of Finite Differences, as its name implies, deals with *finite* increments of the variable or argument and the corresponding increments of the function dependent on the variable.

Let $f(x)$ be a function of the variable x, and let $x_0, x_1, x_2, \ldots$ be the successive values of x, such that the intervals

$$x_1 - x_0,\ \ x_2 - x_1,\ \ x_3 - x_2,\ \ldots\ldots\ldots$$

are finite. These intervals may or may not be equal. When these are equal, to Δx say, the successive values of x can be put as

$$x,\ \ x+\Delta x,\ \ x+2\Delta x,\ \ x+3\Delta x,\ \ldots\ldots\ldots$$

and the corresponding values of the function as

$$f(x),\ \ f(x+\Delta x),\ \ f(x+2\Delta x),\ \ f(x+3\Delta x),\ \ldots\ldots\ldots$$

The difference of $f(x)$ from its successive value, *i.e.*,

$$f(x+\Delta x) - f(x) \tag{1}$$

is called the first difference of $f(x)$. We shall be mainly concerned with such differences. In most cases the interval Δx will be taken to be unity.

When the values of x increase by unequal finite quantities, the intervals of the variable x are also unequal. We have such cases in Divided Differences, Reciprocal Differences, ...

If the intervals of x become infinitesimally small, we come to Differential Calculus. Thus from (1) we have

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

which in the limit as $\Delta x \longrightarrow 0$ gives the differential coefficient of $f(x)$. Thus

$$f'(x)=\underset{\Delta x\to 0}{Lim}\frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

1.2. Difference Notation. Let Δx be the increment of the variable x, then the corresponding value of the function is $f(x+\Delta x)$. The difference

$$f(x+\Delta x)-f(x)$$

is denoted by $\Delta f(x)$ and is called the *First Difference of* $f(x)$ with respect to the increment Δx. Here we shall be concerned with finite constant increment.

Hence denoting Δx by ω, we have

$$\Delta f(x)=f(x+\omega)-f(x)$$

as the first difference of $f(x)$. Similarly

$$\Delta^2 f(x)=\Delta f(x+\omega)-\Delta f(x)$$

is called the *Second Difference of* $f(x)$, and

$$\Delta^3 f(x)=\Delta^2 f(x+\omega)-\Delta^2 f(x)$$

is the *Third Difference of* $f(x)$. Proceeding in this manner we can form the fourth, fifth......nth differences of $f(x)$.

Consider for example $f(x)=x^3$, then

$$\begin{aligned}\Delta x^3 &=3x^2\omega+3x\omega^2+\omega^3,\\ \Delta^2 x^3 &=6x\omega^2+6\omega^3,\\ \Delta^3 x^3 &=6\omega^3,\\ \Delta^4 x^3 &=0.\end{aligned}$$

We shall now construct a Difference Table corresponding to the increasing values of the argument x from $x=a$.

Argument	Function	Δ	Δ^2	Δ^3	Δ^4
a	$f(a)$				
		$\Delta f(a)$			
$a+\omega$	$f(a+\omega)$		$\Delta^2 f(a)$		
		$\Delta f(a+\omega)$		$\Delta^3 f(a)$	
$a+2\omega$	$f(a+2\omega)$		$\Delta^2 f(a+\omega)$		$\Delta^4 f(a)$
		$\Delta f(a+2\omega)$		$\Delta^3 f(a+\omega)$	
$a+3\omega$	$f(a+3\omega)$		$\Delta^2 f(a+2\omega)$		
		$\Delta f(a+3\omega)$			
$a+4\omega$	$f(a+4\omega)$				

and similarly for differences of orders higher than the fourth. The first entry $f(a)$ is called the *leading term*, $f(a+\omega)$, $f(a+2\omega)$, ... are called the *successive values* of the function, and the differences $\Delta f(a)$, $\Delta^2 f(a)$, $\Delta^3 f(a)$, ... are called the *leading differences*. Each difference in the Table is obtained by subtracting the upper entry immediately to the left from the next lower entry in the same vertical column. It may be noted that to form a fourth difference, five consecutive entries are required; and generally to form a difference of the nth order, $(n+1)$ consecutive entries are necessary.

Example. Difference table of $f(x)=x^3$, $\omega=1$.

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		
		91			
6	216				

1.3. Operator E. This operator denotes the operation of increasing the argument by ω. Thus

$$Ef(x)=f(x+\omega).$$

The operation can be repeated any number of times. Thus

$$E^2f(x)=Ef(x+\omega)=f(x+2\omega),$$

$$E^3f(x)=E^2f(x+\omega)=f(x+3\omega),$$

……………………………………

$$E^nf(x)=E^{n-1}f(x+\omega)=f(x+n\omega),$$

where n is an integer. We have by § 1·2

$$\Delta f(x)=f(x+\omega)-f(x)=Ef(x)-f(x)$$
$$=(E-1)f(x).$$

Also $$f(x+\omega)=f(x)+\Delta f(x).$$

It is thus evident that the operators E and Δ are connected by the relations

$$(1) \qquad \Delta\equiv E-1, \text{ or } E\equiv 1+\Delta.$$

Again by Taylor's Theorem, we have

$$Ef(x)=f(x+\omega)$$
$$=f(x)+\omega\frac{d}{dx}f(x)+\frac{\omega^2}{2!}\frac{d^2}{dx^2}f(x)+\frac{\omega^3}{3!}\frac{d^3}{dx^3}f(x)+\ldots$$
$$=e^{\omega\frac{d}{dx}}f(x).$$

Hence $$E\equiv e^{\omega\frac{d}{dx}}\equiv e^{\omega D}$$

where D stands for $\frac{d}{dx}$.

In most cases we shall take ω to be unity. Thus

$$\Delta f(x)=(E-1)f(x)=f(x+1)-f(x)$$

and $$E\equiv 1+\Delta\equiv e^D.$$

1.4. Properties of the Operators. The operators Δ, E and D follow the fundamental laws of Algebra.* We have

$$\begin{aligned}\Delta\{f(x)+g(x)+\ldots\ldots\} \\ &=f(x+\omega)+g(x+\omega)+\ldots\ldots-f(x)-g(x)-\ldots\ldots \\ &=\Delta f(x)+\Delta g(x)+\ldots\ldots\end{aligned}$$

Thus Δ satisfies the *Distributive Law*.

$$\begin{aligned}\text{Again}\quad \Delta^m\Delta^n f(x)&=(\Delta\Delta\ldots m\text{ times})(\Delta\Delta\ldots n\text{ times})\,f(x) \\ &=(\Delta\Delta\ldots\ldots\overline{m+n}\text{ times})\,f(x) \\ &=\Delta^{m+n}f(x).\end{aligned}$$

Thus Δ satisfies the *Index Law*.

Finally $\qquad \Delta af(x)=af(x+\omega)-af(x)=a\Delta f(x),$

where a is constant. This result is also true if a is replaced by a periodic function $a(x)$ of period ω. Thus

$$\begin{aligned}\Delta a(x)f(x)&=a(x+\omega)f(x+\omega)-a(x)f(x) \\ &=a(x)f(x+\omega)-a(x)f(x) \\ &=a(x)\Delta f(x).\end{aligned}$$

Let Δ_x operate on x and Δ_y on y, and let ω and ω' be the respective increments of x and y, then the *partial differences* of $f(x, y)$ are given by

$$\begin{aligned}\Delta_x f(x, y)&=f(x+\omega, y)-f(x, y), \\ \Delta_y f(x, y)&=f(x, y+\omega')-f(x, y).\end{aligned}$$

$$\begin{aligned}\text{Again}\quad \Delta_y\Delta_x f(x, y)&=\Delta_y\{f(x+\omega, y)-f(x, y)\} \\ &=f(x+\omega, y+\omega')-f(x+\omega, y)-f(x, y+\omega')+f(x, y). \\ \Delta_x\Delta_y f(x, y)&=\Delta_x\{f(x, y+\omega')-f(x, y)\} \\ &=f(x+\omega, y+\omega')-f(x+\omega, y)-f(x, y+\omega')+f(x, y).\end{aligned}$$

Hence $\qquad \Delta_x\Delta_y=\Delta_y\Delta_x.$

Thus the operator Δ satisfies the *Commutative Law* with regard to constants and periodic functions of the same period and to the operators not affecting the argument on which Δ operates.

It may be shewn in like manner that E and D also satisfy the fundamental laws of Algebra subject to the same conditions. Thus in many respects the operators Δ, E and D behave like Algebraic symbols and may be combined as such.

1.5. Differences of Elementary Functions

(1) $\Delta f(x)g(x)=f(x+\omega)g(x+\omega)-f(x)g(x)$

$$=f(x+\omega)g(x+\omega)-f(x)g(x+\omega)+f(x)g(x+\omega)-f(x)g(x)$$

$$=g(x+\omega)\Delta f(x)+f(x)\Delta g(x).$$

Putting $\omega=1$, we have

$$\Delta f(x)g(x)=g(x+1)\Delta f(x)+f(x)\Delta g(x).$$

Again $\Delta f(x)g(x)=\{f(x)+\Delta f(x)\}\{g(x)+\Delta g(x)\}-f(x)g(x)$

$$=f(x)\Delta g(x)+g(x)\Delta f(x)+\Delta f(x)\Delta g(x).$$

(2) $\Delta\dfrac{f(x)}{g(x)}=\dfrac{f(x+\omega)}{g(x+\omega)}-\dfrac{f(x)}{g(x)}$

$$=\frac{\{f(x)+\Delta f(x)\}g(x)-\{g(x)+\Delta g(x)\}f(x)}{g(x)g(x+\omega)}$$

$$=\frac{g(x)\Delta f(x)-f(x)\Delta g(x)}{g(x)g(x+\omega)}.$$

(3)* $\Delta\sin u_x=\sin(u_x+\Delta u_x)-\sin u_x$

$$=2\sin\frac{\Delta u_x}{2}\cos\left(u_x+\frac{\Delta u_x}{2}\right)$$

$$=2\sin\frac{\Delta u_x}{2}\sin\left(u_x+\frac{\pi+\Delta u_x}{2}\right).$$

$$\Delta\cos u_x=2\sin\frac{\Delta u_x}{2}\cos\left(u_x+\frac{\pi+\Delta u_x}{2}\right).$$

$$\Delta\tan u_x=\tan u_{x+\omega}-\tan u_x=\frac{\sin\Delta u_x}{\cos u_x\cos u_{x+\omega}}.$$

$$\Delta\tan^{-1}u_x=\tan^{-1}u_{x+\omega}-\tan^{-1}u_x=\tan^{-1}\frac{\Delta u_x}{1+u_xu_{x+\omega}}.$$

---

* For the sake of convenience we denote the function $u(x)$ by u_x.

If $u_x = ax+b$, and $\omega=1$, we have

$$\Delta \sin(ax+b) = 2\sin\tfrac{a}{2}\sin\left(ax+b+\frac{a+\pi}{2}\right).$$

Similarly $$\Delta^2 \sin(ax+b) = (2\sin\tfrac{a}{2})^2 \sin\left(ax+b+2.\frac{a+\pi}{2}\right),$$

and generally $$\Delta^n \sin(ax+b) = (2\sin\tfrac{a}{2})^n \sin\left(ax+b+n.\frac{a+\pi}{2}\right).$$

Similarly $$\Delta \cos(ax+b) = 2\sin\tfrac{a}{2}\cos\left(ax+b+\frac{a+\pi}{2}\right),$$

and $$\Delta^n \cos(ax+b) = (2\sin\tfrac{a}{2})^n \cos\left(ax+b+n.\frac{a+\pi}{2}\right).$$

$$\Delta \tan(ax+b) = \frac{\sin a}{\cos(ax+b)\cos(ax+a+b)}$$

$$\Delta \tan^{-1}(ax+b) = \tan^{-1}\frac{a}{1+(ax+b)(ax+a+b)}.$$

(4) $$\Delta a^x = a^{x+\omega} - a^x = (a^\omega - 1)a^x.$$

Hence $$\Delta^2 a^x = (a^\omega - 1)^2 a^x,$$

and generally $$\Delta^n a^x = (a^\omega - 1)^n a^x.$$

If $\omega=1$, we have $\Delta^n a^x = (a-1)^n a^x$.

Again $$\Delta a^{mx} = (a^m - 1)a^{mx},$$

$$\Delta^n a^{mx} = (a^m - 1)^n a^{mx}.$$

(5) $\Delta \log u_x = \log u_{x+\omega} - \log u_x$

$$= \log\frac{u_{x+\omega}}{u_x} = \log\left(1+\frac{\Delta u_x}{u_x}\right) = \frac{\Delta u_x}{u_x} - \tfrac{1}{2}\left(\frac{\Delta u_x}{u_x}\right)^2 + \tfrac{1}{3}\left(\frac{\Delta u_x}{u_x}\right)^3 - \ldots$$

Also $$\Delta \log(u_x u_{x-1} \ldots\ldots u_{x-m+1}) = \log\frac{u_{x+1}}{u_{x-m+1}}.$$

1.6. Differences of a Polynomial. Consider the polynomial

$$f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots\ldots+a_{n-1}x+a_n.$$

Now $\Delta f(x)=f(x+\omega)-f(x)$

$$=a_0\{(x+\omega)^n-x^n\}+a_1\{(x+\omega)^{n-1}-x^{n-1}\}+\ldots+a_{n-1}\omega$$

$$=na_0x^{n-1}\omega+\left\{\frac{n(n-1)}{2!}a_0\omega^2+(n-1)a_1\omega\right\}x^{n-2}+\ldots\ldots+a_{n-1}\omega.$$

This is a polynomial of degree $n-1$ in x and therefore the first difference of a polynomial represents another polynomial of degree less by one unit. Repeating this process, we have

$$\Delta^2 f(x)=a_0n(n-1)x^{n-2}\omega^2+\ldots\ldots$$

a polynomial of degree $n-2$ and so on. If $r<n$, we obtain on differencing r times

$$\Delta^r f(x)=a_0n(n-1)(n-2)\ldots\ldots(n-r+1)x^{n-r}\omega^r+\ldots\ldots$$

Finally $\Delta^n f(x)=a_0n(n-1)(n-2)\ldots\ldots2.1.\omega^n.$

Putting $\omega=1$, we have $\Delta^n f(x)=a_0n(n-1)(n-2)\ldots\ldots2.1.$

In particular $\Delta^n x^n=1.2.3\ldots\ldots.n.$

Thus the nth difference of a polynomial of degree n is constant and the $(n+1)$th and higher differences are all zeroes.

1.7. Differences of Factorial Expressions. Products of the form

$$f(x)f(x-\omega)f(x-2\omega)\ldots\ldots f(x-m\omega+\omega)$$

and
$$f(x)f(x+\omega)f(x+2\omega)\ldots\ldots f(x+m\omega-\omega)$$

where m is a positive integer, are called *Factorial* expressions, the first being a descending factorial and the second an ascending factorial expression.

The simplest and of special interest are the following factorials:

$x(x-1(x-2)\ldots\ldots(x-m+1)$ represented by $x^{(m)}$

and $\dfrac{1}{x(x+1)(x+2)\ldots\ldots(x+m-1)}$ represented by $x^{(-m)}$.

If $m=0$, the value of these expressions is unity, *i.e.*,

$$x^{(0)}=x^{(-0)}=1.$$

Now $\Delta x^{(m)} = (x+1)x(x-1)\ldots\ldots(x-m+2)$

$$-x(x-1)(x-2)\ldots\ldots(x-m+1)$$

$$=mx(x-1)(x-2)\ldots\ldots(x-m+2)$$

$$=mx^{(m-1)}=m^{(1)}x^{(m-1)}.$$

Again $\quad \Delta^2 x^{(m)} = m\Delta x^{(m-1)} = m(m-1)x^{(m-2)} = m^{(2)}x^{(m-2)}$

and generally $\Delta^n x^{(m)} = m(m-1)\ldots\ldots(m-n+1)x^{(m-n)} = m^{(n)}x^{(m-n)}.$

It may be noted that the sum of prefixes of m and x is always equal to m and thus the number of factors in m and x is always m.

Again $\quad \Delta x^{(-m)} = \dfrac{1}{(x+1)(x+2)\ldots(x+m)} - \dfrac{1}{x(x+1)\ldots(x+m-1)}$

$$=\frac{-m}{x(x+1)(x+2)\ldots(x+m)}$$

$$=-mx^{(-m-1)}=(-m)^{(1)}x^{(-m-1)}.$$

Hence by repeated operations of Δ we have

$$\Delta^n x^{(-m)} = -m(-m-1)\ldots\ldots(-m-n+1)x^{(-m-n)}$$

$$=(-m)^{(n)}x^{(-m-n)}$$

$$=(-1)^n m(m+1)(m+2)\ldots\ldots(m+n-1)x^{(-m-n)}.$$

Here the number of factors in the denominator exceeds the number of factors in the numerator by m.

The two cases can symbolically be put in the single form

$$\Delta^n x^{(m)} = m^{(n)} x^{(m-n)}$$

where m may be a positive or negative integer.

For the general form of factorials we have

$$\Delta u_x u_{x-1}\ldots\ldots u_{x-m+1} = (u_{x+1} - u_{x-m+1})u_x u_{x-1}\ldots\ldots u_{x-m+2}$$

and $\quad \Delta \dfrac{1}{u_x u_{x+1}\ldots u_{x+m-1}} = \dfrac{u_x - u_{x+m}}{u_x u_{x+1}\ldots u_{x+m}}.$

If $u_x = ax+b$, we have

$$\Delta(ax+b)^{(m)} = am(ax+b)^{(m-1)},$$

$$\Delta^2(ax+b)^{(m)} = a^2m(m-1)(ax+b)^{(m-2)},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Delta^n(ax+b)^{(m)} = a^n m^{(n)}(ax+b)^{(m-n)}.$$

If $m=n$, $$\Delta^n(ax+b)^{(m)} = a^n m\,!$$

and if $m < n$, $$\Delta^n(ax+b)^{(m)} = 0.$$

Again $$\Delta(ax+b)^{(-m)} = -am(ax+b)^{(-m-1)},$$

$$\Delta^n(ax+b)^{(-m)} = (-1)^n a^n m(m+1)\ldots\ldots(m+n-1)(ax+b)^{(-m-n)}.$$

For the interval ω we have

$$x^{(m\omega)} = x(x-\omega)\,(x-2\omega)\ldots\ldots(x-m\omega+\omega),$$

and $$x^{(-m\omega)} = \frac{1}{x(x+\omega)(x+2\omega)\ldots\ldots(x+m\omega-\omega)}.$$

Hence $$\Delta x^{(m\omega)} = m\omega x^{(m\omega-\omega)},$$

$$\Delta^n x^{(m\omega)} = m(m-1)\ldots\ldots(m-n+1)\omega^n x^{(m\omega-n\omega)},$$

$$\Delta x^{(-m\omega)} = -m\omega x^{(-m\omega-\omega)},$$

$$\Delta^n x^{(-m\omega)} = (-m)\,(-m-1)\ldots(-m-n+1)\omega^n x^{(-m\omega-n\omega)}.$$

1.8. Difference Quotient $\underset{\omega}{\Delta}$. We introduce here Nörlund's operator $\underset{\omega}{\Delta}$ which is defined by the relation

$$\underset{\omega}{\Delta} f(x) = \frac{f(x+\omega)-f(x)}{\omega}.$$

We call $\underset{\omega}{\Delta} f(x)$ the *first difference quotient* of $f(x)$. The operation can be repeated. Thus

$$\underset{\omega}{\Delta}^2 f(x) = \underset{\omega}{\Delta}\left\{\underset{\omega}{\Delta} f(x)\right\} = \frac{\underset{\omega}{\Delta} f(x+\omega) - \underset{\omega}{\Delta} f(x)}{\omega}$$

$$= \frac{f(x+2\omega)-2f(x+\omega)+f(x)}{\omega^2}.$$

And generally for the nth difference quotient

$$\underset{\omega}{\Delta}^n f(x)=\underset{\omega}{\Delta}\left\{\underset{\omega}{\Delta}^{n-1}f(x)\right\}.$$

We obtain from this the relation

$$\underset{\omega}{\Delta}^n f(x+\omega)=\omega\underset{\omega}{\Delta}^{n+1}f(x)+\underset{\omega}{\Delta}^n f(x).$$

The operators Δ and $\underset{\omega}{\Delta}$ are connected by the relation

$$\Delta^n=\omega^n.\underset{\omega}{\Delta}^n.$$

When $\omega=1$, both the operators become the same.

When $\omega\longrightarrow 0$, we have

$$\lim_{\omega\to 0}\underset{\omega}{\Delta}f(x)=Df(x)$$

and

$$\lim_{\omega\to 0}\underset{\omega}{\Delta}^n f(x)=D^n f(x),\qquad \text{where } D=\frac{d}{dx}.$$

1.9. Gregory-Newton Formula. *Expression of $u_{x+n\omega}$ in terms of u_x and its successive differences.*

We have $u_{x+n\omega}=E^n u_x=(1+\Delta)^n u_x$

$$=\left\{1+n\Delta+\frac{n(n-1)}{2!}\Delta^2+\frac{n(n-1)(n-2)}{3!}\Delta^3+\ldots+\Delta^n\right\}u_x.$$

Hence

(1) $$u_{x+n\omega}=u_x+n\Delta u_x+\frac{n(n-1)}{2!}\Delta^2 u_x+\ldots+\Delta^n u_x.$$

Example. Express x^n in descending and ascending factorials.

We have by Gregory-Newton formula

$$u_{x+y}=u_y+x\Delta u_y+\frac{x(x-1)}{2!}\Delta^2 u_y+\ldots+\Delta^x u_y.$$

Put $u_x=x^n$, then

$$(x+y)^n=y^n+x\Delta y^n+\frac{x(x-1)}{2!}\Delta^2 y^n+\ldots\ldots+\Delta^x y^n.$$

Putting $y=0$, we have

$$x^n=0^n+x\Delta 0^n+\frac{x(x-1)}{2!}\Delta^2 0^n+\ldots\ldots+\Delta^x 0^n$$

giving x^n in descending factorials.

To express x^n in ascending factorials put $-x$ for x and let $u_x=x^{n+1}$. Then

$$(y-x)^{n+1}=y^{n+1}-x\Delta y^{n+1}+\frac{x(x+1)}{2!}\Delta^2 y^{n+1}$$
$$-\frac{x(x+1)(x+2)}{3!}\Delta^3 y^{n+1}+\dots$$

Putting $y=0$, we have

$$(-x)^{n+1}=-x\Delta 0^{n+1}+\frac{x(x+1)}{2!}\Delta^2 0^{n+1}-\frac{x(x+1)(x+2)}{3!}\Delta^3 0^{n+1}+\dots$$

$$\text{or } x^n=(-1)^n\left\{\Delta 0^{n+1}-\frac{x+1}{2!}\Delta^2 0^{n+1}+\frac{(x+1)(x+2)}{3!}\Delta^3 0^{n+1}-\dots\right\}.$$

1.10. *Expression for $\Delta^n u_x$ in terms of u_x and its successive values.*

$$\Delta^n u_x=(E-1)^n u_x$$
$$=\left\{E^n-nE^{n-1}+\frac{n(n-1)}{2!}E^{n-2}-\dots+(-1)^n\right\}u_x$$
$$=u_{x+n\omega}-nu_{x+\overline{n-1}\,\omega}+\frac{n(n-1)}{2!}u_{x+\overline{n-2}\,\omega}+\dots+(-1)^n u_x$$

When $\omega=1$, we have

$$\Delta^n u_x=u_{x+n}-nu_{x+n-1}+\frac{n(n-1)}{2!}u_{x+n-2}-\dots+(-1)^n u_x.$$

In particular, putting $u_x=x^m$, we have

$$\Delta^n x^m=(x+n)^m-n(x+n-1)^m+\frac{n(n-1)}{2!}(x+n-2)^m-\dots+(-1)^n x^m.$$

Let $x=0$, then

$$\Delta^n 0^m=n^m-n(n-1)^m+\frac{n(n-1)}{2!}(n-2)^m+\dots+(-1)^n 0^m.$$

Similarly $\Delta^n 0^n=n^n-n(n-1)^n+\frac{n(n-1)}{2!}(n-2)^n-\dots+(-1)^n 0^n.$

But by § 1.6 we have $\Delta^n 0^n=n!$. Hence

$$1.\,2.\,3\dots\dots n=n^n-n(n-1)^n+\frac{n(n-1)}{2!}(n-2)^n-\dots\dots+(-1)^n 0^n.$$

1.11. Theorem. *If $F(t)$ be a rational integral function of t with coefficients which are independent of t, then will*

(1) $$F(E)a^x\phi(x)=a^xF(a^\omega E)\phi(x).$$

Let $$F(t)=\Sigma a_n t^n,$$

then
$$\begin{aligned}F(E)a^x\phi(x)&=\Sigma a_n E^n a^x\phi(x)\\&=\Sigma a_n a^{x+n\omega}\phi(x+n\omega)\\&=a^x\Sigma a_n a^{n\omega}\phi(x+n\omega)\\&=a^x\Sigma a_n(a^\omega E)^n\phi(x)\\&=a^xF(a^\omega E)\phi(x).\end{aligned}$$

In particular when $\omega=1$, we have

(2) $$F(E)a^x\phi(x)=a^xF(aE)\phi(x).$$

From this we easily have

(3) $$F(E)\phi(x)=a^xF(aE)a^{-x}\phi(x).$$

1.12. Expansion of a Polynomial in Factorials. Let $\phi(x)$ be a polynomial of degree m. To express it in factorials assume

(1) $$\phi(x)=a_0+a_1x+a_2x(x-\omega)+a_3x(x-\omega)(x-2\omega)+\dots\dots$$
$$+a_mx(x-\omega)(x-2\omega)\dots(x-m\omega+\omega)$$

which is a polynomial of degree m with $m+1$ constants. Differencing it we have

$$\Delta\phi(x)=a_1\omega+2a_2\omega x+3a_3\omega x(x-\omega)+\dots+ma_m\omega x^{(\overline{m-1}\,\omega)}$$

or $$\underset{\omega}{\Delta}\phi(x)=a_1+2a_2x+3a_3x^{(2\omega)}+\dots+ma_mx^{(\overline{m-1}\,\omega)}.$$

Similarly $$\underset{\omega}{\Delta}^2\phi(x)=2a_2+3.2.a_3x+\dots+m(m-1)a_mx^{(\overline{m-2}\,\omega)},$$

$$\dots\dots\dots\dots$$

$$\underset{\omega}{\Delta}^r\phi(x)=r(r-1)\dots2.1.a_r+(r+1)r\dots2a_{r+1}x+\dots+m^{(r)}a_mx^{(\overline{m-r}\,\omega)},$$

$$\dots\dots\dots\dots$$

$$\underset{\omega}{\Delta}^m\phi(x)=m(m-1)\dots.1.a_m.$$

Now putting $x=0$ in the above equations we obtain

$$\phi(0)=a_0,\quad \underset{\omega}{\Delta}\phi(0)=a_1,\quad \underset{\omega}{\Delta}^2\phi(0)=2a_2,\ \dots$$

$$\underset{\omega}{\Delta}^r\phi(0)=r!\ a_r,\dots\underset{\omega}{\Delta}^m\phi(0)=m!\ a_m.$$

Substituting these values of $a_0,\ a_1,\ a_2,\ \dots$ in (1) we have

$$(2)\quad \phi(x)=\phi(0)+x\underset{\omega}{\Delta}\phi(0)+\frac{x^{(2\ \omega)}}{2!}\underset{\omega}{\Delta}^2\phi(0)+\dots\dots$$

$$+\frac{x^{(r\ \omega)}}{r!}\underset{\omega}{\Delta}^r\phi(0)+\dots\dots\dots+\frac{x^{(m\ \omega)}}{m!}\underset{\omega}{\Delta}^m\phi(0).$$

When $\omega\longrightarrow 0$, we get Maclaurin's Theorem in Differential Calculus

$$(3)\qquad \phi(x)=\phi(0)+xD\phi(0)+\frac{x^2}{2!}D^2\phi(0)+\dots\dots+\frac{x^m}{m!}D^m\phi(0).$$

When $\omega=1$, we have

$$(4)\qquad \phi(x)=\phi(0)+x\Delta\phi(0)+\frac{x^{(2)}}{2!}\Delta^2\phi(0)+\dots\dots+\frac{x^{(m)}}{m!}\Delta^m\phi(0).$$

1.13. Secondary Form of Maclaurin's Theorem. *To expand* $\phi(x)$ *in the form*

$$\phi(x)=\phi(0)+\phi\left(\frac{d}{d0}\right)0.x+\phi\left(\frac{d}{d0}\right)\frac{0^2.x^2}{2!}+\phi\left(\frac{d}{d0}\right)\frac{0^3.x^3}{3!}+\dots$$

where $\phi\left(\frac{d}{d0}\right)0^m$ *denotes the value of* $\phi\left(\frac{d}{dx}\right)x^m$ *when* $x=0$.

To establish this result we shall first prove the lemma that if $\phi(x)$ and $\psi(x)$ be two rational integral functions of x, then will

$$\phi\left(\frac{d}{d0}\right)\psi(0)=\psi\left(\frac{d}{d0}\right)\phi(0).$$

Let $\qquad \phi(x)=a_0+a_1x+a_2x^2+\dots+a_mx^m+\dots+a_nx^n+\dots$

and $\qquad \psi(x)=b_0+b_1x+b_2x^2+\dots+b_mx^m+\dots+b_nx^n+\dots$

Then $$\phi\left(\frac{d}{dx}\right)\psi(x)=\Sigma a_m b_n\left(\frac{d}{dx}\right)^m x^n$$

and $$\psi\left(\frac{d}{dx}\right)\phi(x)=\Sigma a_m b_n\left(\frac{d}{dx}\right)^n x^m.$$

Now $$\left(\frac{d}{dx}\right)^m x^n=0, \quad \text{if } m>n,$$

$$=0, \quad \text{if } m<n \text{ and } x=0,$$

$$=n\,!, \text{ if } m=n.$$

Hence $$\left\{\left(\frac{d}{dx}\right)^m x^n\right\}_{x=0}=\left\{\left(\frac{d}{dx}\right)^n x^m\right\}_{x=0}.$$

Thus the corresponding terms of the two series are equal and hence

(1) $$\phi\left(\frac{d}{d0}\right)\psi(0)=\psi\left(\frac{d}{d0}\right)\phi(0).$$

Now Maclaurin's theorem (3) of § 1.12 can be put as

(2) $$\phi(x)=\phi(0)+\frac{d}{d0}\phi(0).x+\frac{d^2}{d0^2}\phi(0).\frac{x^2}{2!}+\dots$$

Applying the lemma of reciprocity (1) we get the secondary form of Maclaurin

(3) $$\phi(x)=\phi(0)+\phi\left(\frac{d}{d0}\right).0.x+\phi\left(\frac{d}{d0}\right).\frac{0^2.x^2}{2!}+\dots$$

Since $\phi(D).1=\phi(0)$, the result (3) can be put as

(4) $$\phi(x)=\phi(D)e^{0\cdot x}.$$

The forms (2) and (3) of Maclaurin's Theorem may be termed *Conjugate*.

1.14. Herschel's Theorem. *If $f(x)$ be a rational integral function of x, then*

$$f(e^t)=f(E)e^{0\cdot t}=f(1)+f(E)0.t+f(E)\frac{0^2.t^2}{2!}+\ldots.$$

Assume $\quad f(e^t)=a_0+a_1e^t+a_2e^{2t}+a_3e^{3t}+\ldots$

This can be put as

$$a_0e^{xt}+a_1e^{(x+1)t}+a_2e^{(x+2)t}+\ldots\ldots \qquad \text{where } x=0.$$

Hence $f(e^t)=\{(a_0+a_1E+a_2E^2+\ldots)e^{xt}\}_{x=0}$

$$=\{f(E)e^{xt}\}_{x=0}$$

$$=f(E)e^{0\cdot t}$$

$$=\Sigma f(E)\frac{0^m.t^m}{m!}.$$

Now $f(E).1=f(1).1$. Hence

$$f(e^t)=f(1)+f(E)0.t+f(E)\frac{0^2.t^2}{2!}+\ldots$$

1.15. Deduction of Maclaurin's Theorem from Herschel's Theorem, and conversely.

We have

$$\phi(t)=\phi(\log e^t)=\phi(\log E)e^{0\cdot t}, \text{ by Herschel's theorem}$$

$$=\phi(\log e^D)e^{0\cdot t}=\phi(D)e^{0\cdot t},$$

which is the secondary form of Maclaurin's theorem.

To establish the converse case, put $f(e^t)=\phi(t)$.

Then $\phi(0)=f(1)$ and $\phi\left(\frac{d}{d0}\right)=f\left(e^{\frac{d}{d0}}\right)=f(E)$.

Thus $f(e^t)=\phi(t)=\phi(0)+\phi\left(\frac{d}{d0}\right)0.t+\phi\left(\frac{d}{d0}\right)\frac{0^2.t^2}{2!}+\ldots$

by Maclaurin's theorem

$$=f(1)+f(E)0.t+f(E)\frac{0^2.t^2}{2!}+\ldots$$

which is Herschel's theorem.

Example 1. Develop in ascending powers of t

$$(i)\quad \frac{1}{1+e^t}, \qquad (ii)\quad \frac{t}{e^t-1}.$$

(*i*) Here $f(e^t) = \dfrac{1}{1+e^t} = \dfrac{1}{1+E}\, e^{0 \cdot t}$, by Herschel's theorem

$$= \frac{1}{2+\Delta}\, e^{0 \cdot t} = \tfrac{1}{2}\left(1+\frac{\Delta}{2}\right)^{-1} . \, e^{0 \cdot t}$$

$$= \tfrac{1}{2}\left\{1-\frac{\Delta}{2}+\frac{\Delta^2}{2^2}-\frac{\Delta^3}{2^3}+\ldots\ldots+(-1)^x\frac{\Delta^x}{2^x}+\ldots\right\} e^{0 \cdot t}$$

$$= \tfrac{1}{2}\left\{1-\frac{\Delta}{2}+\frac{\Delta^2}{2^2}-\frac{\Delta^3}{2^3}+\ldots+(-1)^x\frac{\Delta^x}{2^x}+\ldots\right\}$$

$$\times\left\{1+0.t+\frac{0^2.t^2}{2!}+\ldots+\frac{0^x.t^x}{x!}+\ldots\right\}.$$

Hence $f(e^t) = \Sigma\, \tfrac{1}{2}\left\{1-\dfrac{\Delta}{2}+\dfrac{\Delta^2}{2^2}-\ldots+(-1)^x\dfrac{\Delta^x}{2^x}\right\}\dfrac{0^x.t^x}{x!}$,

i.e., $\dfrac{1}{1+e^t} = \Sigma\, \tfrac{1}{2}\left\{0^x-\dfrac{\Delta 0^x}{2}+\dfrac{\Delta^2 0^x}{2^2}-\ldots+(-1)^x\dfrac{\Delta^x 0^x}{2^x}\right\}\dfrac{t^x}{x!}$.

(*ii*) $\dfrac{t}{e^t-1} = \dfrac{\log e^t}{e^t-1} = \dfrac{\log E}{E-1}\, e^{0 \cdot t}$, by Herschel's theorem

$$= \frac{\log(1+\Delta)}{\Delta}\, e^{0 \cdot t}$$

$$= \left\{1-\frac{\Delta}{2}+\frac{\Delta^2}{3}-\frac{\Delta^3}{4}+\ldots+(-1)^x\frac{\Delta^x}{x+1}-\ldots\right\} e^{0 \cdot t}$$

$$= \Sigma\left\{0^x-\frac{\Delta 0^x}{2}+\frac{\Delta^2 0^x}{3}-\ldots+(-1)^x\frac{\Delta^x 0^x}{x+1}\right\}\frac{t^x}{x!}.$$

Example 2. Shew that

$$e^{e^t} = e\left\{1+e^{\Delta}0.\,t+e^{\Delta}\frac{0^2.t^2}{2!}+\ldots\ldots\right\}$$

and calculate the first four terms of the expression.

[*C. U. M.A. & M.Sc. 1938.*]

We have

$$f(e^t)=e^{e^t}=e^{1+\Delta}.e^{0\cdot t}, \text{ by Herschel's theorem}$$

$$=e.e^{\Delta}\left(1+0.t+\frac{0^2.t^2}{2!}+\ldots+\frac{0^x.t^x}{x!}+\ldots\right)$$

$$=e.\left\{e^{\Delta}1+e^{\Delta}0.t+e^{\Delta}\frac{0^2.t^2}{2!}+\ldots+e^{\Delta}\frac{0^x.t^x}{x!}+\ldots\right\}$$

$$=e\left\{1+e^{\Delta}0.t+e^{\Delta}\frac{0^2.t^2}{2!}+\ldots+e^{\Delta}\frac{0^x.t^x}{x!}+\ldots\right\},$$

giving the result of the first portion.

$$\text{Now } f(e^t)=e.e^{\Delta}\left(1+0.t+\frac{0^2.t^2}{2!}+\ldots+\frac{0^x.t^x}{x!}+\ldots\right)$$

$$=e\left\{1+\Delta+\frac{\Delta^2}{2!}+\ldots+\frac{\Delta^x}{x!}+\ldots\right\}$$

$$\times\left\{1+0.t+\frac{0^2.t^2}{2!}+\ldots\frac{0^x.t^x}{x!}+\ldots\right\}$$

$$=\Sigma\frac{et^x}{x!}\left\{0^x+\frac{\Delta 0^x}{1}+\frac{\Delta^2 0^x}{2!}+\ldots+\frac{\Delta^x 0^x}{x!}\right\}.$$

Putting $x=0, 1, 2$ and 3, we have

$$e^{e^t}=e+et+et^2+\tfrac{5}{6}et^3+\ldots$$

1.16. Lagrange's Theorem. *To prove that*

$$\Delta^n u_x=\frac{d^n u_x}{dx^n}+\frac{\Delta^n 0^{n+1}}{(n+1)!}.\frac{d^{n+1}u_x}{dx^{n+1}}+\frac{\Delta^n 0^{n+2}}{(n+2)!}.\frac{d^{n+2}u_x}{dx^{n+2}}+\ldots$$

where n is a positive integer.

We have $\Delta^n u_x=(E-1)^n u_x=(e^D-1)^n u_x$.

Now by Herschel's theorem

$$(e^t-1)^n=(E-1)^n e^{0\cdot t}=\Delta^n e^{0\cdot t}$$

$$=\Delta^n\left\{1+0.t+\frac{0^2.t^2}{2!}+\dots+\frac{0^n.t^n}{n!}+\frac{0^{n+1}.t^{n+1}}{(n+1)!}+\dots\right\}.$$

But $\Delta^n 0^m=0$ if $n>m$, and $=n!$ if $n=m$. Hence

$$(e^t-1)^n=t^n+\frac{\Delta^n 0^{n+1}}{(n+1)!}.t^{n+1}+\frac{\Delta^n 0^{n+2}}{(n+2)!}.t^{n+2}+\dots$$

Replacing t by D we have

(1) $$\Delta^n u_x=\frac{d^n u_x}{dx^n}+\frac{\Delta^n 0^{n+1}}{(n+1)!}\cdot\frac{d^{n+1}u_x}{dx^{n+1}}+\frac{\Delta^n 0^{n+2}}{(n+2)!}\cdot\frac{d^{n+2}u_x}{dx^{n+2}}+\dots$$

1.17. Generalised Lagrange's Theorem.

We have by Taylor's theorem

$$u_{x+t}=u_x+tDu_x+\frac{t^2}{2!}D^2u_x+\dots\dots$$

Operating by $\phi(\Delta)$ on both sides (Δ operating on t only) we have

$$\phi(\Delta)u_{x+t}=\phi(\Delta)u_x+\phi(\Delta)t.\frac{du_x}{dx}+\phi(\Delta)\frac{t^2}{2!}\cdot\frac{d^2u_x}{dx^2}+\dots\dots$$

The first term $\phi(\Delta)u_x=u_x\phi(\Delta).1=u_x\phi(0).1.$

Now putting $t=0$, we have

$$\phi(\Delta)u_x=\phi(0)u_x+\phi(\Delta)0.\frac{du_x}{dx}+\phi(\Delta)\frac{0^2}{2!}\cdot\frac{d^2u_x}{dx^2}+\dots\dots$$

Putting $\phi(\Delta)=\Delta^n$, we get Lagrange's theorem, §1.16(1).

1.18. Expressions for $\Delta^n u_x v_x$.

We have $\Delta u_x v_x=u_{x+1}v_{x+1}-u_xv_x$

$$=Eu_xE'v_x-u_xv_x$$

where E operates on u_x alone and E' on v_x alone. Then

$$\Delta u_x v_x = (EE' - 1) u_x v_x.$$

Hence $\Delta^n u_x v_x = (EE' - 1)^n u_x v_x$

$$= (-1)^n \left\{ u_x v_x - n u_{x+1} v_{x+1} \right.$$

$$\left. + \frac{n(n-1)}{2!} u_{x+2} v_{x+2} - \dots + (-1)^n u_{x+n} v_{x+n} \right\}.$$

Again to express it in ascending differences of v_x, we have

$$\Delta^n u_x v_x = \{E(1+\Delta') - 1\}^n u_x v_x = \{\Delta + E\Delta'\}^n u_x v_x$$

$$= \left\{ \Delta^n + n\Delta^{n-1} E\Delta' + \frac{n(n-1)}{2!} \Delta^{n-2} E^2 \Delta'^2 + \dots \right\} u_x v_x$$

$$= \Delta^n u_x . v_x + n\Delta^{n-1} u_{x+1} . \Delta v_x + \frac{n(n-1)}{2!} \Delta^{n-2} u_{x+2} . \Delta^2 v_x + \dots$$

(we have dropped the accents of Δ' as no longer necessary).

Putting $u_x = a^x$, we have

$$\Delta^n a^x v_x = a^x \left\{ (a-1)^n v_x + n(a-1)^{n-1} a \Delta v_x \right.$$

$$\left. + \frac{n(n-1)}{2!} (a-1)^{n-2} a^2 \Delta^2 v_x + \dots \right\}.$$

1.19. Expressions for Du_x and $\phi(D) u_x$ in terms of differences of u_x.

We have $e^D = 1 + \Delta$, hence $D = \log(1+\Delta)$.

$$\therefore \quad Du_x = \log(1+\Delta) u_x$$

$$= \left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots \right) u_x$$

$$= \Delta u_x - \frac{\Delta^2 u_x}{2} + \frac{\Delta^3 u_x}{3} - \frac{\Delta^4 u_x}{4} + \dots\dots$$

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Similarly $D^n u_x = \left\{\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots\right\}^n u_x.$

Writing $-n$ for n we have

$$\iiint \dots (dx)^n u_x = \{\log(1+\Delta)\}^{-n} u_x.$$

Again $\quad u_{x+t} = u_x + t\Delta u_x + \frac{t(t-1)}{2!}\Delta^2 u_x + \dots\dots$

Operating by $\phi\left(\frac{d}{dt}\right)$ on both sides we have

$$\phi\left(\frac{d}{dt}\right) u_{x+t} = \phi\left(\frac{d}{dt}\right) u_x + \phi\left(\frac{d}{dt}\right) t.\Delta u_x$$

$$+\phi\left(\frac{d}{dt}\right)\frac{t(t-1)}{2!}.\Delta^2 u_x + \dots\dots$$

Putting $t=0$ and noting that

$$\left\{\phi\left(\frac{d}{dt}\right) u_{x+t}\right\}_{t=0} = \phi\left(\frac{d}{dx}\right) u_x, \quad \phi\left(\frac{d}{d0}\right).1 = \phi(0).1$$

we have

$$\phi\left(\frac{d}{dx}\right) u_x = u_x \phi(0) + \phi\left(\frac{d}{d0}\right) 0.\,\Delta u_x + \phi\left(\frac{d}{d0}\right) 0^{(2)}.\,\frac{\Delta^2 u_x}{2!} + \dots\dots$$

of which the above are particular cases.

1.20. Differences of Nothing. The differences of the various positive integral powers of x, in which the increment of x is taken to be unity and x is put equal to zero after evaluation, are known as *Differences of Zero* or *Differences of Nothing*. They are represented by the symbol $\Delta^m 0^n$. To evaluate them, x^n is differenced m times and then x is put equal to zero.

They are of frequent occurrence and play an important part in the Calculus of Finite Differences. We establish here a formula connecting differences of nothing, from which we are

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able to construct a table of such differences. We shall first establish the formula

$$\phi(E)0^n = E\phi'(E)0^{n-1}.$$

Let $a_r E^r 0^n$ and $Era_r E^{r-1}0^{n-1}$ be the corresponding terms of the two sides. Then, since each is equivalent to $a_r . r^n$, the above identity is established.

Again since $E=1+\Delta$, we have

$$\phi(1+\Delta)0^n = E\phi'(1+\Delta)0^{n-1}$$

or

$$f(\Delta)0^n = Ef'(\Delta)0^{n-1}.$$

Putting $f(\Delta)=\Delta^m$, we have at once the recurrence formula

(1) $$\Delta^m 0^n = E.m\Delta^{m-1}0^{n-1} = m(\Delta^{m-1}0^{n-1} + \Delta^m 0^{n-1}).$$

It is to be noted that $\Delta^0 0^1 = 0$ and $\Delta^1 0^1 = 1$

and $$\Delta^m 0^n = n! \quad \text{for} \quad m=n$$
$$= 0 \quad \text{for} \quad m>n.$$

The above formula (1) can also be obtained from the expansion of $\Delta^m x^n$ of §1.10. We have

(2) $$\Delta^m x^n = (x+m)^n - m(x+m-1)^n + \frac{m(m-1)}{2!}(x+m-2)^n - \ldots \ldots\ldots + (-1)^m x^n.$$

Putting $x=0$, we get

$$\Delta^m 0^n = m^n - m(m-1)^n + \frac{m(m-1)}{2!}(m-2)^n - \ldots + (-1)^m 0^n$$

$$= m\left\{m^{n-1} - (m-1)^n + \frac{(m-1)}{2!}(m-2)^n - \ldots + (-1)^{m-1}\right\}$$

$$= m\Delta^{m-1}1^{n-1} \text{ from (2).}$$

Hence $$\Delta^m 0^n = m\Delta^{m-1}1^{n-1} = m\Delta^{m-1}E.0^{n-1}$$

$$= m(\Delta^{m-1}0^{n-1} + \Delta^m 0^{n-1}).$$

We give below a table of differences of nothing:

n	$\Delta 0^n$	$\Delta^2 0^n$	$\Delta^3 0^n$	$\Delta^4 0^n$	$\Delta^5 0^n$	$\Delta^6 0^n$
1	1					
2	1	2				
3	1	6	6			
4	1	14	36	24		
5	1	30	150	240	120	
6	1	62	540	1560	1800	720
7	1	126	1806	8400	16800	15120

To find the value of a particular difference $\Delta^m 0^n$, add two numbers in the preceding row which are situated in the same column and in the preceding column, and multiply by m, *e.g.*,

$$\Delta^4 0^6 = 4(240+150) = 1560.$$

1.21. Difference Quotients of Zero. When the interval is ω, the expression $\underset{\omega}{\Delta}^m 0^n$ is called the *Difference Quotient of Zero*.

Now $\Delta^m x^n = \Delta^m x^{n-1}.x = x\Delta^m x^{n-1} + m\omega\Delta^{m-1}(x+\omega)^{n-1}$, by §1.18.

$\therefore \quad \Delta^m 0^n = m\omega\Delta^{m-1}E0^{n-1} = m\omega(\Delta^m 0^{n-1} + \Delta^{m-1}0^{n-1})$.

Hence we have the recurrence formula for difference quotients

(1) $$\underset{\omega}{\Delta}^m 0^n = m\omega\underset{\omega}{\Delta}^m 0^{n-1} + m\underset{\omega}{\Delta}^{m-1}0^{n-1}.$$

Putting $\omega = 1$, we have $\Delta^m 0^n = m(\Delta^m 0^{n-1} + \Delta^{m-1}0^{n-1})$.

1.22. Differences of Powers of Unity. We establish here the recurrence formula for calculating the differences of powers of unity:

$$\Delta^n 1^{p+1} = (n+1)\Delta^n 1^p + n\Delta^{n-1}1^p.$$

We have
$$\begin{aligned}\Delta^n 1^{p+1} &= \Delta^n E0^{p+1} = \Delta^n(1+\Delta)0^{p+1}\\ &= \Delta^{n+1}0^{p+1} + \Delta^n 0^{p+1}\\ &= (n+1)\{\Delta^{n+1}0^p + \Delta^n 0^p\}\\ &\qquad + n\{\Delta^n 0^p + \Delta^{n-1}0^p\}, \text{ by §1.20 (1)}\\ &= (n+1)\Delta^n E0^p + n\Delta^{n-1}E0^p.\end{aligned}$$

Hence $$\Delta^n 1^{p+1} = (n+1)\Delta^n 1^p + n\Delta^{n-1}1^p.$$

Example. Shew that $\Delta^{n-1}2^p = \dfrac{\Delta^n 1^{p+1} + \Delta^{n-1}1^p}{n+1}$.

We have
$$\begin{aligned}\Delta^n 1^{p+1} + \Delta^{n-1}1^p &= (n+1)\Delta^n 1^p + n\Delta^{n-1}1^p + \Delta^{n-1}1^p\\ &= (n+1)\{\Delta^n 1^p + \Delta^{n-1}1^p\}\\ &= (n+1)\Delta^{n-1}E.1^p\\ &= (n+1)\Delta^{n-1}E(x^p)_{x=1}\\ &= (n+1)\Delta^{n-1}2^p.\end{aligned}$$

Thus
$$\Delta^{n-1}2^p = \frac{\Delta^n 1^{p+1} + \Delta^{n-1}1^p}{n+1}.$$

1.23. Generating Functions. If $\phi(t)$ can be expanded in the form of a series $\Sigma u_x t^x$, then $\phi(t)$ is called the *Generating Function of* u_x and is represented in the form
$$\phi(t) = Gu_x.$$

For example, we have $e^t = \Sigma \dfrac{t^x}{x!} = G.\dfrac{1}{1.2.3\ldots x}$.

Similarly
$$\frac{e^t}{t} = \Sigma \frac{t^x}{(x+1)!} = G.\frac{1}{1.2.3\ldots.(x+1)},$$

$$\frac{e^t}{t^2} = \Sigma \frac{t^x}{(x+2)!} = G.\frac{1}{1.2.3\ldots.(x+2)}.$$

In general
$$\phi(t) = \Sigma u_x t^x = Gu_x,$$

hence
$$\frac{\phi(t)}{t} = \Sigma u_{x+1} t^x = Gu_{x+1},$$

$$\frac{\phi(t)}{t^2} = \Sigma u_{x+2} t^x = Gu_{x+2},$$

(1)
$$\frac{\phi(t)}{t^n} = \Sigma u_{x+n} t^x = Gu_{x+n}.$$

Again $Gu_{x+1} - Gu_x = \left(\frac{1}{t} - 1\right)\phi(t)$

or
$$G\Delta u_x = \left(\frac{1}{t} - 1\right)\phi(t).$$

$$\begin{aligned} G\Delta^2 u_x &= G\Delta(u_{x+1} - u_x) \\ &= G(u_{x+2} - 2u_{x+1} + u_x) \\ &= \left(\frac{1}{t} - 1\right)^2 \phi(t). \end{aligned}$$

$$G\Delta^n u_x = \left(\frac{1}{t} - 1\right)^n \phi(t). \tag{2}$$

Again

$$\begin{aligned} Gu_{x+n} &= \left(\frac{1}{t}\right)^n \phi(t) = \left\{1 + \left(\frac{1}{t} - 1\right)\right\}^n \phi(t) \\ &= \phi(t) + n\left(\frac{1}{t} - 1\right)\phi(t) + \frac{n(n-1)}{2!}\left(\frac{1}{t} - 1\right)^2 \phi(t) + \dots \\ &= Gu_x + nG\Delta u_x + \frac{n(n-1)}{2!} G\Delta^2 u_x + \dots \\ &= G\left\{u_x + n\Delta u_x + \frac{n(n-1)}{2!}\Delta^2 u_x + \dots\right\}. \end{aligned}$$

Hence we have the expression for u_{x+n} already obtained in §1.9.

Also
$$\begin{aligned} t\phi(t) &= \Sigma u_{x-1} t^x = Gu_{x-1}, \\ t^2\phi(t) &= \Sigma u_{x-2} t^x = Gu_{x-2}, \end{aligned}$$

$$t^n \phi(t) = \Sigma u_{x-n} t^x = Gu_{x-n}. \tag{3}$$

$$Gu_x - Gu_{x-1} = (1-t)\phi(t),$$

or
$$G\Delta u_{x-1} = (1-t)\phi(t),$$

$$\begin{aligned} G\Delta^2 u_{x-2} &= Gu_x - 2Gu_{x-1} + Gu_{x-2} \\ &= (1-t)^2 \phi(t). \end{aligned}$$

$$G\Delta^n u_{x-n} = (1-t)^n \phi(t). \tag{4}$$

1.24. Theorem. *If* $\phi(t)=Gu_x$ *and* $\psi(t)=\Sigma a_x t^x$, *then will*

$$\phi(t)\psi(t)=G\psi\left(\frac{1}{E}\right)u_x$$

and
$$\phi(t)\psi\left(\frac{1}{t}\right)=G\psi(E)u_x.$$

The coefficient of t^x in $\phi(t)\psi(t)$

$$=\ldots+a_{-1}u_{x+1}+a_0u_x+a_1u_{x-1}+a_2u_{x-2}+a_3u_{x-3}+\ldots$$
$$=(\ldots+a_{-1}E+a_0+a_1E^{-1}+a_2E^{-2}+a_3E^{-3}+\ldots)u_x$$
$$=\psi\left(\frac{1}{E}\right)u_x.$$

Hence
$$\phi(t)\psi(t)=G\psi\left(\frac{1}{E}\right)u_x.$$

The coefficient of t^x in $\phi(t)\psi\left(\frac{1}{t}\right)$

$$=\ldots+a_{-1}u_{x-1}+a_0u_x+a_1u_{x+1}+a_2u_{x+2}+\ldots$$
$$=\psi(E)u_x.$$

Hence
$$\phi(t)\psi\left(\frac{1}{t}\right)=G\psi(E)u_x.$$

The formulæ of the previous article are all obtained by giving particular values to $\psi(E)$. Thus taking $\psi(E)=E^n$,

we have
$$\psi\left(\frac{1}{t}\right)=\frac{1}{t^n}$$

and thus
$$\frac{\phi(t)}{t^n}=Gu_{x+n}.$$

Again for
$$\psi(E)=\frac{1}{E^n},\ \psi\left(\frac{1}{t}\right)=t^n$$

and thus
$$t^n\phi(t)=Gu_{x-n}.$$

Putting $\psi(E)=\Delta^n=(E-1)^n$, $\psi\left(\frac{1}{t}\right)=\left(\frac{1}{t}-1\right)^n$

and hence $$G\Delta^n u_x=\left(\frac{1}{t}-1\right)^n Gu_x.$$

Putting $\psi(E)=\left(\frac{\Delta}{E}\right)^n=\left(1-\frac{1}{E}\right)^n$. $\psi\left(\frac{1}{t}\right)=(1-t)^n$

and hence $$G\Delta^n u_{x-n}=(1-t)^n\phi(t).$$

Example 1. **Shew that**

$$u_{x+n}=u_x+n\Delta u_{x-1}+\frac{n(n+1)}{2!}\Delta^2 u_{x-2}+\frac{n(n+1)(n+2)}{3!}\Delta^3 u_{x-3}+\dots$$

We have by §1.23(1)

$$Gu_{x+n}=\frac{1}{t^n}\phi(t)=\{1-(1-t)\}^{-n}\phi(t)$$

$$=\phi(t)+n(1-t)\phi(t)+\frac{n(n+1)}{2!}(1-t)^2\phi(t)+\frac{n(n+1)(n+2)}{3!}(1-t)^3\phi(t)+\dots$$

Hence by §1.23(4) we have

$$Gu_{x+n}=Gu_x+nG\Delta u_{x-1}+\frac{n(n+1)}{2!}G\Delta^2 u_{x-2}$$
$$+\frac{n(n+1)(n+2)}{3!}G\Delta^3 u_{x-3}+\dots$$

whence the result.

Example 2. **Shew that**

$$u_{x+n}=u_x+n\Delta u_{x-r}+\frac{n(n+2r-1)}{2!}\Delta^2 u_{x-2r}$$
$$+\frac{n(n+3r-1)(n+3r-2)}{3!}\Delta^3 u_{x-3r}+\dots$$

We have by §1.23(3) $t^r\phi(t)=Gu_{x-r}$.

Hence by §1.24

$$t^r\left(\frac{1}{t}-1\right)\phi(t)=\left(\frac{1}{t}-1\right)Gu_{x-r}=G\Delta u_{x-r}.$$

Similarly $t^{2r}\left(\frac{1}{t}-1\right)^2\phi(t)=G\Delta^2u_{x-2r}$, and so on.

Here we shall have to expand Gu_{x+n} in a series whose terms are of the form

$$\left\{t^r\left(\frac{1}{t}-1\right)\right\}^k\phi(t), \qquad k=1, 2, 3\ldots$$

Put $t^r\left(\frac{1}{t}-1\right)=z$, then $\frac{1}{t}-1=\frac{z}{t^r}$.

or $\frac{1}{t}=1+\frac{z}{t^r}$, or $y=1+zy^r$ where $y=\frac{1}{t}$.

By Lagrange's theorem, if $y=u+z\phi(y)$, then we have

$$f(y)=f(u)+z\phi(u)f'(u)+\frac{z^2}{2!}\frac{d}{du}\left\{[\phi(u)]^2f'(u)\right\}+\ldots$$

Taking $f(y)=y^n$, $\phi(u)=u^r$ and $u=1$, we have

$$y^n=1+nz+\frac{n(n+2r-1)}{2!}z^2+\frac{n(n+3r-1)(n+3r-2)}{3!}z^3+\ldots$$

Substituting the values for y and z we have

$$\frac{1}{t^n}=1+n\left\{t^r\left(\frac{1}{t}-1\right)\right\}+\frac{n(n+2r-1)}{2!}\left\{t^r\left(\frac{1}{t}-1\right)\right\}^2$$

$$+\frac{n(n+3r-1)(n+3r-2)}{3!}\left\{t^r\left(\frac{1}{t}-1\right)\right\}^3+\ldots$$

Hence

$$GU_{x+n}=GU_x+nG\Delta U_{x-r}+\frac{n(n+2r-1)}{2!}G\Delta^2U_{x-2r}$$

$$+\frac{n(n+3r-1)(n+3r-2)}{3!}G\Delta^3U_{x-3r}+\ldots$$

whence the result.

MISCELLANEOUS EXAMPLES

Example 1. Sum to infinity

$$\Delta x^m - \tfrac{1}{2}\Delta^2 x^m + \frac{1.3}{2.4}\Delta^3 x^m - \frac{1.3.5}{2.4.6}.\Delta^4 x^m + \dots$$

[*C. U. M.A. & M.Sc. 1938.*]

$$\text{The series } = \Delta\left\{1 - \tfrac{1}{2}\Delta + \frac{1.3}{2.4}\Delta^2 - \frac{1.3.5}{2.4.6}\Delta^3 + \dots\right\}x^m$$

$$= \Delta\{(1+\Delta)^{-\frac{1}{2}}\}x^m$$

$$= \Delta E^{-\frac{1}{2}}x^m$$

$$= (x+\tfrac{1}{2})^m - (x-\tfrac{1}{2})^m.$$

Example 2. Shew that

$$\int_0^\infty e^{-nx}(e^{-x}-1)^m x^{a-1}dx = \Gamma(a)\Delta^m n^{-a},$$

n and a being positive quantities and Δ operating on n alone.

[*C. U. M.A. & M.Sc. 1936. 1938.*]

Since $\quad \Delta^m e^{-nx} = (e^{-x}-1)^m e^{-nx}$

we have $$\int_0^\infty e^{-nx}(e^{-x}-1)^m x^{a-1}dx = \int_0^\infty \Delta^m e^{-nx}.x^{a-1}dx$$

$$= \Delta^m \int_0^\infty e^{-nx}x^{a-1}dx.$$

Put $nx = y$, then

$$\int_0^\infty e^{-nx}x^{a-1}dx = \int_0^\infty e^{-y}y^{a-1}n^{-a}dy$$

$$= \Gamma(a)n^{-a}$$

for $$\int_0^\infty e^{-y}y^{a-1}dy = \Gamma(a).$$

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Hence $$\int_0^\infty e^{-nx}(e^{-x}-1)^m x^{a-1}dx=\Delta^m\Gamma(a)n^{-a}$$

$$=\Gamma(a)\Delta^m n^{-a}.$$

Example 3. If $x=e^\theta$, establish the formula

$$\left(\frac{d}{d\theta}\right)^n=\frac{\Delta 0^n}{1}x\frac{d}{dx}+\frac{\Delta^2 0^n}{2\,!}x^2\frac{d^2}{dx^2}+\frac{\Delta^3 0^n}{3\,!}x^3\frac{d^3}{dx^3}+\dots$$

[*C. U. M.A. & M.Sc. 1933, 1935.*]

Since $x=e^\theta$, we have $x\dfrac{d}{dx}=\dfrac{d}{d\theta}$.

Also, by a well-known formula in Differential Calculus,

$$x^n\frac{d^n}{dx^n}=\frac{d}{d\theta}\left(\frac{d}{d\theta}-1\right)\left(\frac{d}{d\theta}-2\right)\dots\dots\left(\frac{d}{d\theta}-\overline{n-1}\right).$$

Thus $$x^2\frac{d^2}{dx^2}=\frac{d}{d\theta}\left(\frac{d}{d\theta}-1\right)$$

$$x^3\frac{d^3}{dx^3}=\frac{d}{d\theta}\left(\frac{d}{d\theta}-1\right)\left(\frac{d}{d\theta}-2\right)$$

……………………………………

Now

$$\frac{\Delta 0^n}{1}x\frac{d}{dx}+\frac{\Delta^2 0^n}{2\,!}x^2\frac{d^2}{dx^2}+\frac{\Delta^3 0^n}{3\,!}x^3\frac{d^3}{dx^3}+\dots$$

$$=0^n+\left\{\Delta x\frac{d}{dx}+\frac{\Delta^2}{2\,!}x^2\frac{d^2}{dx^2}+\frac{\Delta^3}{3\,!}x^3\frac{d^3}{dx^3}+\dots\right\}0^n$$

$$=\left\{1+\Delta\frac{d}{d\theta}+\frac{\Delta^2}{2\,!}\frac{d}{d\theta}\left(\frac{d}{d\theta}-1\right)\right.$$

$$\left.+\frac{\Delta^3}{3\,!}\frac{d}{d\theta}\left(\frac{d}{d\theta}-1\right)\left(\frac{d}{d\theta}-2\right)+\dots\right\}0^n$$

$$=\left\{(1+\Delta)^{\frac{d}{d\theta}}\right\}0^n=E^{\frac{d}{d\theta}}.0^n=\left(0+\frac{d}{d\theta}\right)^n=\left(\frac{d}{d\theta}\right)^n.$$

Example 4. Shew that

$$\Delta^m 0^n + n\Delta^m 0^{n-1} + \frac{n(n-1)}{2!}\Delta^m 0^{n-2} + \ldots\ldots + \frac{n!}{(n-m)!} = \frac{\Delta^{m+1}0^{n+1}}{m+1}.$$

Now

$$\Delta x^n = (x+1)^n - x^n = nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2} + \frac{n(n-1)(n-2)}{3!}x^{n-3} + \ldots$$

Hence $\Delta 0^n = n0^{n-1} + \frac{n(n-1)}{2!}0^{n-2} + \frac{n(n-1)(n-2)}{3!}0^{n-3} + \ldots$

Also $\quad \Delta^{m+1}0^{n+1} = (m+1)(\Delta^m 0^n + \Delta^{m+1}0^n)$.

The last term $= \frac{n!}{(n-m)!} = \frac{n(n-1)(n-2)\ldots(m+1)}{(n-m)!}\Delta^m 0^m$.

$\therefore$ The given expression

$$= \Delta^m 0^n + \Delta^m \left\{ n0^{n-1} + \frac{n(n-1)}{2!}0^{n-2} + \frac{n(n-1)(n-2)}{3!}0^{n-3} + \ldots + \frac{n(n-1)(n-2)\ldots(m+1)}{(n-m)!}0^m \right\}$$

$$= \Delta^m 0^n + \Delta^m \Delta 0^n$$

$$= \Delta^m 0^n + \Delta^{m+1} 0^n$$

$$= \frac{\Delta^{m+1}0^{n+1}}{m+1}.$$

EXAMPLES

1. Find the first differences of the following functions :—

(*i*) $\tan x\theta$. (*ii*) $\cot x\theta$. (*iii*) $2^x \sin \frac{\theta}{2^x}$.

(*iv*) $\left(2^x \sin \frac{\theta}{2^x}\right)^2$. (*v*) $\tan \frac{\theta}{2^x}$. (*vi*) $\cot 2^x\theta$.

(*vii*) $2^x \tan \frac{\theta}{2^x}$. (*viii*) $\frac{1}{2^x \sin \frac{\theta}{2^x}}$. (*ix*) $\frac{1}{2^x \tan \frac{\theta}{2^x}}$.

(*x*) $\dfrac{1}{\left(2^x\tan\dfrac{\theta}{2^x}\right)^2}$. (*xi*) $\dfrac{\cos\dfrac{\theta}{2^x}}{\left(2^x\sin\dfrac{\theta}{2^x}\right)^3}$. (*xii*) $\dfrac{1}{\left(2^x\sin\dfrac{\theta}{2^x}\right)^2}$.

(*xiii*) $\dfrac{\log(2\sin 2^x\theta)}{2^x}$. (*xiv*) $2^x\tan^{-1}\dfrac{\theta}{2^x}$. (*xv*) $\dfrac{a^{2^x}+1}{a^{2^x}-1}$.

(*xvi*) $2^x.\dfrac{a^{2^x}+1}{a^{2^x}-1}$. (*xvii*) $\tan^{-1}\dfrac{\alpha x+\beta}{ax+b}$.

2. **Prove the following relations :—**

(*i*) $0^x-\Delta 0^x+\Delta^2 0^x-\ldots\pm\Delta^x 0^x=(-1)^x.$

(*ii*) $\dfrac{\Delta 0^x}{1}-\dfrac{\Delta^2 0^x}{2}+\ldots\pm\dfrac{\Delta^x 0^x}{x}=0.$

3. **Prove the following identities :**

(*i*) $\Delta^n 0^{n+1}=\dfrac{n(n+1)}{2}\Delta^n 0^n.$

(*ii*) $(n+1)\Delta^n 0^n=2\{\Delta^{n-1}0^n+\Delta^n 0^n\}.$

4. **Establish**

$$f\{(1+\Delta)^n\}0^x=n^x f(1+\Delta)0^x.$$

5. **Shew that if** m **be less than** r,

$$\{1+\log E\}^r 0^m=r(r-1)(r-2)\ldots(r-m+1).$$

[*C. U. M.A. & M.Sc. 1936.*]

6. **Shew that**

$$\{\log(1+\Delta)\}^n f(\Delta)0^x=x(x-1)(x-2)\ldots(x-n+1)f(\Delta)0^{x-n}.$$

7. **Prove that**

$$\{\log E\}^n 0^m=0,$$

unless $m=n$, when it is equal to $n!$.

8. **Shew that**

$$f(\Delta)0^{n+1}=(n+1)\frac{f(\Delta)}{\log(1+\Delta)}0^n.$$

[*C. U. M.A. & M.Sc. 1934.*]

9. Prove that

$$\left\{f(1+\Delta)+f\left(\frac{1}{1+\Delta}\right)\right\}0^{2x-1}=0,$$

and

$$\left\{f(1+\Delta)-f\left(\frac{1}{1+\Delta}\right)\right\}0^{2x}=0,$$

and thence deduce the expression

$$\frac{\Delta 0^{2x}}{2}-\frac{\Delta^2 0^{2x}}{2^2}+\frac{\Delta^3 0^{2x}}{2^3}-\ldots\pm\frac{\Delta^{2x}0^{2x}}{2^{2x}}=0.$$

10. Assuming $f(1+\Delta)=\int\frac{\log(1+\Delta)d\Delta}{\Delta}$, prove that

$$\frac{\Delta 0^{2x}}{1^2}-\frac{\Delta^2 0^{2x}}{2^2}+\ldots\pm\frac{\Delta^{2x}0^{2x}}{(2x)^2}=0$$

in all cases except where $x=1$.

11. Prove that

(i) $$(a+e^t)^n=\Sigma\frac{(1+a)^n t^x}{x!}\left\{0^x+\frac{n\Delta 0^x}{1+a}+\frac{n(n-1)}{2!}\frac{\Delta^2 0^x}{(1+a)^2}+\ldots\right.$$
$$\left.+\frac{n(n-1)\ldots(n-x+1)}{x!}\frac{\Delta^x 0^x}{(1+a)^x}\right\}.$$

(ii) $$\frac{1}{\sqrt{1+e^t}}=\Sigma\frac{t^x}{\sqrt{2}\,x!}\left\{0^x-\frac{1}{4}\Delta 0^x+\frac{1.3}{4.8}\Delta^2 0^x-\ldots\right.$$
$$\left.+(-1)^x\frac{1.3.5\ldots\overline{2x-1}}{4.8\ldots\ldots4x}\Delta^x 0^x\right\}.$$

12. If $P_n=\frac{1}{1.n}+\frac{1}{2.(n-1)}+\ldots+\frac{1}{n.1}$, shew that

$$\{P_1\Delta^2-P_2\Delta^3+P_3\Delta^4-\ldots\}0^m=0,\ \text{if } m>2.$$

[*C. U. M.A. & M.Sc. 1936.*]

13. If $P_x^n=\frac{\Delta^n 0^x}{n!}$, shew that $P_x^n=P_{x-1}^{n-1}+nP_{x-1}^n$.

14. Shew that

(i) $$u_0+u_1x+\frac{u_2x^2}{2!}+\ldots=e^x\left\{u_0+x\Delta u_0+\frac{x^2\Delta^2 u_0}{2!}+\ldots\right\}.$$

(ii) $\Delta^n u_{x-n} = u_x - n u_{x-1} + \frac{n(n-1)}{2!} u_{x-2} - \dots\dots$

(iii) $$u_{x+n} = u_n + x\Delta u_{n-1} + \frac{x(x+1)}{2!}\Delta^2 u_{n-2} + \frac{x(x+1)(x+2)}{3!}\Delta^3 u_{n-3} + \dots\dots$$

(iv) $$u_0 + x\Delta u_1 + \frac{x(x-1)}{2!}\Delta^2 u_2 + \frac{x(x-1)(x-2)}{3!}\Delta^3 u_3 + \dots\dots$$
$$= u_x + x\Delta^2 u_{x-1} + \frac{x(x-1)}{2!}\Delta^4 u_{x-2} + \dots\dots$$

15. Find u_x from the following relations

$$Gu_x = \frac{1-\sqrt{1-4t}}{2t}, \quad Gu_x = f(e^t).$$

16. Shew that $G(\log E)^n u_x = \left(\log\frac{1}{t}\right)^n Gu_x = GD^n u_x$

and $$G \cos x\theta = \frac{t\cos\theta - t^2}{1-2t\cos\theta + t^2}.$$

17. Shew that $B(m+1, n) = (-1)^m \Delta^m \frac{1}{n}$, where m is a positive integer.

18. Shew that

$$u_x = u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + \dots + \Delta^{n-1} u_{x-n} + \Delta^n u_{x-n}$$

[*C. U. M.A. & M.Sc. 1938.*]

19. Sum to n terms

$$1.2\,\Delta x^n - 2.3\,\Delta^2 x^n + 3.4\,\Delta^3 x^n - 4.5\,\Delta^4 x^n + \dots$$

20. Shew that

$$\begin{aligned} \underset{\omega}{\Delta}^n \frac{x^{(m\omega)}}{m!} &= \frac{x^{(m\omega - n\omega)}}{(m-n)!}, && \text{for } n<m, \\ &= 1, && \text{for } n=m, \\ &= 0, && \text{for } n>m. \end{aligned}$$

21. Shew that

$$\underset{\omega}{\Delta}^n (m-1)!\, x^{(-m\omega)} = (-1)^n (m+n-1)!\, x^{(-m\omega - n\omega)}$$

and $$D^n (m-1)!\, x^{-m} = (-1)^n (m+n-1)!\, x^{-m-n}.$$

22. Shew that $\underset{\omega}{\Delta}(ax+b)^{(m\omega)} = am(ax+b)^{(m\omega-\omega)}$,

$$\underset{\omega}{\Delta}(ax+b)^{(-m\omega)} = -am(ax+b)^{(-m\omega-\omega)}.$$

23. Shew that

$$\int_0^\infty \frac{\sin 2nx \sin^m x}{x^{a+1}}\,dx = \frac{\pi\Delta^m(2n-m)^a}{2^{m+1}\Gamma(a+1)\cos\frac{a+m}{2}\pi},$$

where $2n > m > a$, all being positive and Δ operating on n alone.

Hence shew that

$$\int_0^\infty \frac{\sin 2nx \sin^m x}{x^{m+1}}\,dx$$

is constant for all values of n between $\frac{m}{2}$ and ∞.

24. If $u_x = a_1x + a_2x^2 + a_3x^3 + \ldots\ldots$ *ad infinitum*, show that it can also be expressed in the form

$$u_x = \frac{a_1x}{1-x} + \frac{\Delta a_1x^2}{(1-x)^2} + \frac{\Delta^2a_1x^3}{(1-x)^3} + \ldots\ldots$$

Δ operating on a's only.

[*C. U. M.A. & M.Sc. 1939.*]

25. Shew that the differences of zero

$$\Delta^n0^n,\ \Delta^n0^{n+1},\ \Delta^n0^{n+2},\ldots\ldots$$

form a recurring series, and find its scale of relation.

26. Shew that

$$\int_0^\infty \Delta^m\frac{x^{a+1}}{x^2+n^2}\,dx = (-1)^{k+1}.\frac{1}{2}.\frac{\pi}{\sin\frac{1}{2}a\pi}\,\Delta^m n^a,$$

where m is a positive integer greater than a, $2k$ the even integer next greater than $a+1$, and Δ operates on n alone.

27. Prove that

$1^n + 2^nx + 3^nx^2 + \ldots$ *ad infinitum*

$$= \frac{1}{(1-x)^2}\left\{\Delta0^n + \frac{x}{1-x}\Delta^20^n + \frac{x^2}{(1-x)^2}\Delta^30^n + \ldots\right\}.$$

CHAPTER II

FINITE INTEGRATION AND SUMMATION OF SERIES

2.1. Summation Problem. The problem of Integration or Summation consists in determining a function $\phi(x)$ which will satisfy the relation

(1) $$\Delta\phi(x)=u_x,$$

where u_x is a given function of x. Thus $\phi(x)$ is such a function of x that, when it is differenced, it becomes equivalent to u_x. We are to find the value of the unknown function $\phi(x)$ whose difference is known. If we operate by the inverse of Δ on both sides we get

(2) $$\phi(x)=\Delta^{-1}u_x.$$

This operation is known as *Summation*, for, as we shall presently see, it consists in finding the sum of terms like u_x. It is analogous to integration and for that reason it is also called *Integration*. We shall be concerned mostly with *Finite Summation* of series.

From (1) we have $\phi(x+1)-\phi(x)=u_x$.

Hence

(3) $$\begin{cases} \phi(x)-\phi(x-1)=u_{x-1}, \\ \phi(x-1)-\phi(x-2)=u_{x-2}, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \phi(x-n+1)-\phi(x-n)=u_{x-n}, \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{cases}$$

Summing these we get

$$\phi(x)=u_{x-1}+u_{x-2}+u_{x-3}+\dots\dots+u_{x-n}+\dots\dots$$

(4) $$=\sum_{n=1}^{\infty}u_{x-n}.$$

This sum is called the *symbolic solution of (1) to the left*. If the series converge uniformly in any region, it represents an analytic solution of (1).

The equation (1) is also satisfied if we stop at a particular term u_a, $a<x-1$. For taking

$$\phi(x)=u_{x-1}+u_{x-2}+\ldots+u_a$$

we have $$\Delta\phi(x)=u_x+u_{x-1}+\ldots+u_a-(u_{x-1}+u_{x-2}+\ldots+u_a)$$
$$=u_x.$$

Thus $$\phi(x)=\Delta^{-1}u_x=\sum_{t=a}^{x-1}u_t.$$

It is usual to represent the series $\sum_{t=a}^{x-1} u_t$ by the symbol Σu_x. Thus in Finite Calculus

$$\Sigma u_x=u_{x-1}+u_{x-2}+\ldots\ldots+u_a,$$

the series beginning with the term u_{x-1}. Hence

(5) $$\phi(x)=\Delta^{-1}u_x=\Sigma u_x$$
$$=u_{x-1}+u_{x-2}+\ldots\ldots+u_a.$$

Thus Δ and Σ are *inverse* operators. Sometimes the symbol S is used for the inverse operation of Δ. Thus

$$\phi(x)=\Delta^{-1}u_x=\mathsf{S}\,u_x.$$

Again from the general relation (3) we have

$$\phi(x)-\phi(a)=u_{x-1}+u_{x-2}+\ldots\ldots+u_a,$$

or $$\phi(x)=\Delta^{-1}u_x=\Sigma u_x+\phi(a).$$

It may be thought that corresponding to different integral values of a, we shall get different values of $\Delta^{-1}u_x$, so that the number of expressions denoted by Δ^{-1} is infinite. But that is not so, as is evident from the fact that $\phi(a)$ represents the sum of the terms

$$u_{a-1}+u_{a-2}+u_{a-3}+\ldots\ldots$$

The value of $\phi(a)$ does not change when x is increased by unity so that $\Delta\phi(a)=0$. Hence we may regard it as a constant and denote it by C. Thus

$$\phi(x)=\Delta^{-1}u_x=\Sigma u_x+C. \tag{6}$$

This is independent of any particular value of a, and represents the sum of all terms of the series beginning with u_{x-1}. The difference between the forms (5) and (6) is analogous to the difference between the definite integral $\int_a^b f(x)dx$ and the indefinite integral $\int f(x)dx$ in Infinitesimal Calculus.

The expression for Σu_x can also be symbolically obtained thus:

$$\begin{aligned}\Sigma u_x&=\Delta^{-1}u_x=(E-1)^{-1}u_x=E^{-1}\ (1-E^{-1})^{-1}u_x\\&=(E^{-1}+E^{-2}+E^{-3}+\dots\dots)u_x\\&=u_{x-1}+u_{x-2}+\dots+u_a+u_{a-1}+u_{a-2}+\dots\end{aligned}$$

Here Σ stands for the general sum of all the terms. It corresponds to $a=-\infty$. To deduce the general form (6) we must add a constant C and making $C=-\Sigma u_a$ we have the sum for the selected group of terms $u_{x-1}+u_{x-2}+\dots+u_a$.

Another mode of obtaining the sum is as follows. We have

$$\phi(x+1)-\phi(x)=u_x.$$

Hence

$$\left\{\begin{aligned}&\phi(x)-\phi(x+1)=-u_x,\\&\phi(x+1)-\phi(x+2)=-u_{x+1},\\&\phi(x+2)-\phi(x+3)=-u_{x+2},\\&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\&\phi(x+n)-\phi(x+n+1)=-u_{x+n}.\\&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\end{aligned}\right. \tag{7}$$

Adding we have

$$\phi(x)=-u_x-u_{x+1}-u_{x+2}-\dots-u_{x+n}-\dots$$

$$=-\sum_{n=0}^{\infty}u_{x+n}. \tag{8}$$

This sum is termed *the symbolic solution to the right or the principal summation* by Nörlund. If this series converge uniformly in any region, it represents an analytic solution of (1).

2.2. Integrable Forms. We shall not be able to find the sum for any and every form of u_x. Only for certain special forms of u_x, it will be possible to obtain finite expressions for the sum of the series Σu_x, and we give here the most important types of such series. In other cases, the sum will be expressed in the form of infinite series (Chap. III).

I. Factorial Expressions of the form $x^{(m)}$ and $(ax+b)^{(m)}$.

We have by §1.7

$$\Delta x^{(m+1)}=(m+1)x^{(m)}.$$

Hence

$$\Sigma x^{(m)}=\frac{x^{(m+1)}}{m+1}+C, \tag{1}$$

i.e., $\Sigma x(x-1)(x-2)\ldots(x-m+1)=\dfrac{x(x-1)(x-2)\ldots(x-m)}{m+1}+C.$

In this indefinite summation C may be any arbitrary constant. To determine C we shall take Σ within certain prescribed limits. Thus taking the limit between m and n $(n > m)$ we have

$$1.2.3\ldots m+2.3.4\ldots(m+1)+\ldots+(n-m)(n-m+1)\ldots(n-2)(n-1)$$

$$=\left[\frac{x^{(m+1)}}{m+1}+C\right]_m^n=\frac{n(n-1)(n-2)\ldots(n-m)}{m+1}.$$

Or we can take the upper limit only and determine C from the first term. Thus putting n for x we have

$$\Sigma x^{(m)}=\frac{n(n-1)(n-2)\ldots(n-m)}{m+1}+C,$$

and putting $n=m+1$, we get the first term

$$1.2.3\ldots.m=\frac{(m+1)m\ldots2.1}{m+1}+C,$$

so that in this case $C=0$.

Again $\Delta(ax+b)^{(m+1)}=a(m+1)(ax+b)^{(m)}$

$$\therefore \quad \Sigma(ax+b)^{(m)}=\frac{(ax+b)^{(m+1)}}{a(m+1)}+C_1. \tag{2}$$

The operations can be repeated. Thus

$$\Sigma^2(ax+b)^{(m)}=\frac{(ax+b)^{(m+2)}}{a^2(m+1)\,(m+2)}+C_1x+C_2 \tag{3}$$

for $\Sigma C_1=C_1x$.

In general

$$\Sigma^n(ax+b)^{(m)}=\frac{(ax+b)^{(m+n)}}{a^n(m+1)\,(m+2)\ldots(m+n)} \tag{4}$$

$$+C_1x^{n-1}+C_2x^{n-2}+\ldots+C_{n-1}x+C_n.$$

Example. Sum to n terms the series

$$1.3.5+3.5.7+5.7.9+\ldots\ldots$$

Here the nth term $u_n=(2n-1)\,(2n+1)\,(2n+3)$.

Hence the required sum $=\Sigma u_{n+1}$

$$=\Sigma(2n+5)\,(2n+3)\,(2n+1)$$

$$=\frac{(2n+5)\,(2n+3)\,(2n+1)\,(2n-1)}{8}+C.$$

To determine C, put $n=1$, then

$$1.3.5=\frac{7.5.3.1}{8}+C, \quad \text{or} \quad C=\frac{15}{8}.$$

Hence

$$S_n=\frac{(2n-1)\,(2n+1)\,(2n+3)\,(2n+5)+15}{8}.$$

2.3. II. Factorial Expressions of the form $x^{(-m)}$ and $(ax+b)^{(-m)}$

We have by §1.7

$$\Delta x^{(-m+1)}=(-m+1)x^{(-m)}.$$

$$\therefore \quad \Sigma x^{(-m)}=\frac{x^{(-m+1)}}{-m+1}+C. \tag{1}$$

i.e.,

$$\Sigma\frac{1}{x(x+1)\,(x+2)\ldots(x+m-1)}$$

$$=C-\frac{1}{m-1}\cdot\frac{1}{x(x+1)\,(x+2)\ldots(x+m-2)}.$$

Again $\Delta(ax+b)^{(-m+1)}=a(-m+1)(ax+b)^{(-m)}$

(2) $$\therefore \quad \Sigma(ax+b)^{(-m)}=\frac{(ax+b)^{(-m+1)}}{(-1)a(m-1)}+C_1.$$

Repeating the operation we have

(3) $$\Sigma^2(ax+b)^{(-m)}=\frac{(ax+b)^{(-m+2)}}{(-1)^2a^2(m-1)(m-2)}+C_1x+C_2.$$

In general

(4) $$\Sigma^n(ax+b)^{(-m)}=\frac{(ax+b)^{(-m+n)}}{(-1)^n a^n(m-1)(m-2)\ldots(m-n)}$$
$$+C_1x^{n-1}+C_2x^{n-2}+\ldots+C_{n-1}x+C_n.$$

In order that the summation may be possible we must have $m \geq n+1$. Otherwise the denominator in the result will vanish giving no finite expression. For $n=1$, we must have $m \geq 2$, so that there must be at least two factors in the denominator of the function to be integrated once.

It may be noted that these results are all obtained from the results in §2.2 by putting $-m$ for m.

Example. Sum to n terms the series

$$\frac{1}{1.4.7}+\frac{1}{4.7.10}+\frac{1}{7.10.13}+\ldots$$

Here the nth term $=\dfrac{1}{(3n-2)(3n+1)(3n+4)}$.

Hence $$S_n=\Sigma\frac{1}{(3n+1)(3n+4)(3n+7)}$$

$$=C-\frac{1}{3.2.(3n+1)(3n+4)}.$$

Putting $n=1$, we have

$$\frac{1}{1.4.7}=C-\frac{1}{6.4.7}, \text{ or } C=\frac{1}{24}.$$

$$\therefore \quad S_n=\frac{1}{24}-\frac{1}{6(3n+1)(3n+4)}.$$

Working Rule for finding the sum to n terms of $\Sigma(ax+b)^{(m)}$, m being a positive or negative integer (Boole):

Write down the nth term of the series with its factors in ascending order of magnitude, add one factor at the end (take away one factor at the beginning), divide by the number of factors now remaining and by the coefficient of x in each factor, and add to (subtract from) a constant.

The bracketed portions in the above paragraph are to be taken when m is negative.

2.4. III. Polynomials. If the xth term u_x of the series be a polynomial in x, we shall have to transform u_x in a series of factorials by §1.12 and then apply the summation formula of §2.2. By §1.12(4) we have

$$u_x = u_0 + x\Delta u_0 + \frac{x^{(2)}}{2!}\Delta^2 u_0 + \dots$$

(1) $$\therefore \quad \Sigma u_x = C + xu_0 + \frac{x^{(2)}}{2!}\Delta u_0 + \frac{x^{(3)}}{3!}\Delta^2 u_0 + \dots$$

The number of terms in (1) will be finite, for u_x is a rational and integral function of x.

Example 1. Find the sum of n terms of the series

$$1^2 + 2^2 + 3^2 + \dots$$

Here $$\Sigma(n+1)^2 = 1^2 + 2^2 + 3^2 + \dots\dots + n^2,$$

and $$\Delta(n+1)^2 = (n+2)^2 - (n+1)^2 = 2n+3,$$

$$\Delta^2(n+1)^2 = 2.$$

$$\therefore \quad S_n = \Sigma(n+1)^2 = C + n + \frac{n(n-1)}{2!}.3 + \frac{n(n-1)(n-2)}{3!}.2$$

$$= C + \frac{n(n+1)(2n+1)}{6}.$$

Putting $n=1$, we get $C=0$.

A **direct proof** can be given thus:

$$\Sigma u_x = u_0 + u_1 + u_2 + \dots + u_{x-1}$$

$$= (1 + E + E^2 + \dots + E^{x-1})u_0$$

$$= \frac{E^x - 1}{E-1}u_0 = \frac{(1+\Delta)^x - 1}{\Delta}u_0.$$

Therefore

$$\Sigma u_x=\left\{x+\frac{x^{(2)}}{2!}\Delta+\frac{x^{(3)}}{3!}\Delta^2+\dots\right\}u_0. \tag{2}$$

Here the undetermined constant has been avoided and the operations performed on the first term u_0 are direct. We have only to construct the table of differences for the first term. Thus for the above example, we have the table

$$\begin{array}{llccc}
 & & \Delta u_0 & \Delta^2 u_0 & \Delta^3 u_0 \\
1 & u_0=1 & & & \\
 & & 3 & & \\
2 & u_1=4 & & 2 & \\
 & & 5 & & 0 \\
3 & u_2=9 & & 2 & \\
 & & 7 & & \\
4 & u_3=16 & & &
\end{array}$$

Substituting for the values of u_0, Δu_0 and $\Delta^2 u_0$ in (2) we get the same result as already obtained.

Example 2. Find the sum of n terms of the series whose nth term is n^3+7n+4.

$$\begin{array}{lccc}
 & \Delta u_0 & \Delta^2 u_0 & \Delta^3 u_0 \\
u_0=12 & & & \\
 & 14 & & \\
u_1=26 & & 12 & \\
 & 26 & & 6 \\
u_2=52 & & 18 & \\
 & 44 & & \\
u_3=96 & & &
\end{array}$$

Hence $$S_n=12n+\frac{14n(n-1)}{2!}+\frac{12n(n-1)(n-2)}{3!}+\frac{6n(n-1)(n-2)(n-3)}{4!}$$

$$=\frac{1}{4}n(n+2)(n^2+15).$$

2.5. IV. Any Rational Fraction of the form $\dfrac{\phi(x)}{u_x u_{x+1}\dots\dots u_{x+m}}$,

where $u_x=ax+b$, and $\phi(x)$ a polynomial in x of a degree lower by at least two unities than the degree of the denominator.

To sum the series we express $\phi(x)$ in the form

$$\phi(x)=a_0+a_1u_x+a_2u_xu_{x+1}+\ldots\ldots+a_{m-1}u_xu_{x+1}\ldots\ldots u_{x+m-2},$$

where the constant a_0, a_1, a_2, ... are to be determined. Then

$$(1)\qquad \frac{\phi(x)}{u_xu_{x+1}\ldots\ldots u_{x+m}}=\frac{a_0}{u_xu_{x+1}\ldots\ldots u_{x+m}}+\frac{a_1}{u_{x+1}\ldots\ldots u_{x+m}}$$

$$+\ldots\ldots+\frac{a_{m-1}}{u_{x+m-1}u_{x+m}}.$$

Each of the fractions can now be separately integrated.

Again it may happen that some intermediate factors are absent in the denominator. In that case we can supply the missing factors both in the numerator and in the denominator and then the factorial character in the denominator being complete we can apply the above method to obtain the sum.

Example 1. Sum to n terms and to infinity the series

$$\frac{10}{1.2.3.4}+\frac{14}{2.3.4.5}+\frac{18}{3.4.5.6}+\ldots$$

Here the nth term $=\dfrac{4n+6}{n(n+1)(n+2)(n+3)}$

$$=\frac{4}{(n+1)(n+2)(n+3)}+\frac{6}{n(n+1)(n+2)(n+3)}.$$

$$\therefore\quad S_n=C-\frac{4}{2(n+2)(n+3)}-\frac{6}{3(n+1)(n+2)(n+3)}$$

$$=C-\frac{2}{(n+1)(n+3)}.$$

Putting $n=1$, we have $\dfrac{10}{1.2.3.4}=C-\dfrac{2}{2.4}$, whence $C=\dfrac{2}{3}$.

Thus $\quad S_n=\dfrac{2}{3}-\dfrac{2}{(n+1)(n+3)}$, and $S_\infty=\dfrac{2}{3}$

Example 2. Sum to n terms the series

$$\frac{3}{1.2.4}+\frac{4}{2.3.5}+\frac{5}{3.4.6}+\dots$$

[*C. U. M.A. & M.Sc. 1939.*]

Here the nth term $=\dfrac{n+2}{n(n+1)\,(n+3)}$

$$=\frac{(n+2)^2}{n(n+1)\,(n+2)\,(n+3)}$$

$$=\frac{n(n+1)+3n+4}{n(n+1)\,(n+2)\,(n+3)}$$

$$=\frac{1}{(n+2)\,(n+3)}+\frac{3}{(n+1)\,(n+2)\,(n+3)}$$

$$+\frac{4}{n(n+1)\,(n+2)\,(n+3)}.$$

Hence $S_n=C-\dfrac{1}{n+3}-\dfrac{3}{2(n+2)\,(n+3)}-\dfrac{4}{3(n+1)\,(n+2)\,(n+3)}$

$$=C-\frac{6n^2+27n+29}{6(n+1)\,(n+2)\,(n+3)}.$$

Putting $n=1$, we have $\dfrac{3}{1.2.4}=C-\dfrac{6+27+29}{6.2.3.4}$,

whence $$C=\frac{29}{36}.$$

2.6. V. Functions of the form a^x and $a^x\phi(x)$, where $\phi(x)$ is a polynomial.

We have $$\Delta a^x=(a-1)a^x.$$

Hence

$$\Sigma a^x=\frac{a^x}{a-1}+C. \tag{1}$$

In general

$$\Sigma^n a^x=\frac{a^x}{(a-1)^n}+C_1x^{n-1}+C_2x^{n-2}+\dots\dots+C_{n-1}x+C_n. \tag{2}$$

Again
$$\begin{aligned}\Sigma a^x\phi(x) &= \Delta^{-1}a^x\phi(x)\\ &= (E-1)^{-1}a^x\phi(x)\\ &= a^x(aE-1)^{-1}\phi(x), \qquad \text{by §1.11(2)}\\ &= a^x\{a(1+\Delta)-1\}^{-1}\phi(x)\\ &= \frac{a^x}{a-1}\left\{1+\frac{a\Delta}{a-1}\right\}^{-1}\phi(x).\end{aligned}$$

Thus

$$(3)\quad \Sigma a^x\phi(x) = \frac{a^x}{a-1}\left\{\phi(x)-\frac{a}{a-1}\Delta\phi(x)+\frac{a^2}{(a-1)^2}\Delta^2\phi(x)-\dots\right\}+C.$$

Repeating the operations, we obtain an expression for Σ^n. Thus,

$$\Sigma^n a^x\phi(x) = (E-1)^{-n}a^x\phi(x) = \frac{a^x}{(a-1)^n}\left\{1+\frac{a\Delta}{a-1}\right\}^{-n}\phi(x)$$

$$(4)\quad = \frac{a^x}{(a-1)^n}\left\{\phi(x)-\frac{na}{a-1}\Delta\phi(x)+\frac{n(n+1)}{2!}\left(\frac{a}{a-1}\right)^2\Delta^2\phi(x)-\dots\right\}$$
$$+C_1x^{n-1}+C_2x^{n-2}+\dots+C_{n-1}x+C_n.$$

In particular putting $a=-1$, we have

$$(5)\quad \Sigma(-1)^x\phi(x) = \frac{(-1)^{x-1}}{2}\left\{\phi(x)-\frac{1}{2}\Delta\phi(x)+\frac{1}{4}\Delta^2\phi(x)-\dots\right\}+C,$$

and

$$(6)\quad \Sigma^n(-1)^x\phi(x) = \frac{(-1)^{x-n}}{2^n}\left\{\phi(x)-\frac{n}{2}\Delta\phi(x)\right.$$
$$\left.+\frac{n(n+1)}{2!}\cdot\frac{1}{4}\Delta^2\phi(x)-\dots\right\}+C_1x^{n-1}+C_2x^{n-2}+\dots+C_{n-1}x+C_n.$$

The formulæ (5) and (6) enable us to sum the series whose terms are alternately positive and negative.

Example 1. Find the sum to x terms of the series

$$1.3+3.3^2+5.3^3+\dots\dots$$

Here the xth term $\quad u_x=3^x(2x-1)$.

Applying (3) we have

$$S_x = 3^x(2x-1) + \Sigma 3^x(2x-1)$$

$$= 3^x(2x-1) + \frac{3^x}{2}\left\{(2x-1) - \frac{3}{2}.2\right\} + C$$

$$= 3^x(3x-3) + C.$$

Putting $x=1$, we get $C=3$. Thus

$$S_x = 3^{x+1}(x-1) + 3.$$

S_x is also obtained by evaluating $\Sigma u_{x+1} = \Sigma 3^{x+1}(2x+1)$.

Example 2. Find the value of $\Sigma^n 3^{x+1}(2x+1)$.

Applying (4) we have

$$\Sigma^n 3^{x+1}(2x+1) = \frac{3^{x+1}}{2^n}\left\{(2x+1) - n.\frac{3}{2}.2\right\}$$

$$+ C_1 x^{n-1} + C_2 x^{n-2} + \ldots + C_{n-1}x + C_n$$

$$= \frac{3^{x+1}}{2^n}(2x+1-3n) + C_1 x^{n-1} + \ldots + C_{n-1}x + C_n$$

where the C's are to be evaluated.

Putting $n=1$, we obtain the value of S_x.

2.7. VI. Functions of the form $\phi(x)\psi(x)$,

where $\phi(x)$ is a polynomial and $\psi(x)$ is a function of x admitting of repeated integration Σ^n for all integral values of n.

We have

$$\Sigma\phi(x)\psi(x) = (EE'-1)^{-1}\phi(x)\psi(x) = (\Delta + E\Delta')^{-1}\phi(x)\psi(x)$$

where E and Δ operate on $\phi(x)$ only and E' and Δ' on $\psi(x)$ only. Then

$$\Sigma\phi(x)\psi(x) = \frac{1}{E\Delta'}\left(1 + \frac{\Delta}{E\Delta'}\right)^{-1}\phi(x)\psi(x)$$

$$= \frac{1}{E\Delta'}\left\{1 - \frac{\Delta}{E\Delta'} + \frac{\Delta^2}{E^2\Delta'^2} - \ldots\ldots\right\}\phi(x)\psi(x)$$

$$= \phi(x-1)\Delta^{-1}\psi(x) - \Delta\phi(x-2)\Delta^{-2}\psi(x)$$

$$+ \Delta^2\phi(x-3)\Delta^{-3}\psi(x) - \ldots\ldots$$

(dropping the accents of Δ' as no longer necessary)

$$(1) \quad =\phi(x-1)\Sigma\psi(x)-\Delta\phi(x-2)\Sigma^2\psi(x) + \Delta^2\phi(x-3)\Sigma^3\psi(x) - \ldots\ldots + C.$$

We need not add arbitrary constants for each summation Σ on the right side of (1). It will suffice to add one constant C in the result. As $\phi(x)$ is a polynomial the series (1) will terminate.

Again

$$\Sigma^n\phi(x)\psi(x)=\frac{1}{E^n\Delta'^n}\left(1+\frac{\Delta}{E\Delta'}\right)^{-n}\phi(x)\psi(x)$$

$$=\frac{1}{E^n\Delta'^n}\left\{1-n\frac{\Delta}{E\Delta'}+\frac{n(n+1)}{2!}\,\frac{\Delta^2}{E^2\Delta'^2}-\ldots\right\}\phi(x)\psi(x)$$

$$(2) \quad =\phi(x-n)\Sigma^n\psi(x)-n\Delta\phi(x-n-1)\Sigma^{n+1}\psi(x) + \frac{n(n+1)}{2!}\,\Delta^2\phi(x-n-2)\Sigma^{n+2}\psi(x)-\ldots\ldots + C_1x^{n-1}+C_2x^{n-2}+\ldots\ldots+C_{n-1}x+C_n.$$

A slightly different form proceeding by successive differences of $\phi(x)$ can be obtained as follows:

$$\Sigma\phi(x)\psi(x)=(\Delta'+\Delta E')^{-1}\phi(x)\psi(x)$$

$$=\frac{1}{\Delta'}\left\{1+\frac{\Delta E'}{\Delta'}\right\}^{-1}\phi(x)\psi(x)$$

$$=\frac{1}{\Delta'}\left\{1-\frac{\Delta E'}{\Delta'}+\frac{\Delta^2E'^2}{\Delta'^2}-\ldots\ldots\right\}\phi(x)\psi(x)$$

$$(3) \quad =\phi(x)\Sigma\psi(x)-\Delta\phi(x)\Sigma^2\psi(x+1) + \Delta^2\phi(x)\Sigma^3\psi(x+2)-\ldots\ldots+C.$$

Similarly,

$$(4) \quad \Sigma^n\phi(x)\psi(x)=\phi(x)\Sigma^n\psi(x)-n\Delta\phi(x)\Sigma^{n+1}\psi(x+1) + \frac{n(n+1)}{2!}\,\Delta^2\phi(x)\Sigma^{n+2}\psi(x+2)-\ldots\ldots + C_1x^{n-1}+C_2x^{n-2}+\ldots\ldots+C_{n-1}x+C_n.$$

Putting $\psi(x)=a^x$ in (3) and (4) we obtain §2.6(3) and §2.6(4).

Example. Sum to n terms the series whose nth term is $(3n+5)\sin(an+b)$.

Here
$$S_n = (3n+5)\sin(an+b) + \Sigma(3n+5)\sin(an+b)$$
$$= (3n+5)\sin(an+b) + \{3(n-1)+5\}\Sigma\sin(an+b)$$
$$-\Delta\{3(n-2)+5\}\Sigma^2\sin(an+b) + C$$
$$= (3n+5)\sin(an+b) + (3n+2)\frac{\sin\left(an+b-\frac{a+\pi}{2}\right)}{2\sin\frac{a}{2}}$$
$$-\frac{3\sin(an+b-\overline{a+\pi})}{\left(2\sin\frac{a}{2}\right)^2} + C.$$

2.8. Partial Summation. *To prove that*
$$\Sigma(u_x\Delta v_x) = u_xv_x - \Sigma(v_{x+1}\Delta u_x).$$

We have
$$\Delta u_xv_x = u_x\Delta v_x + v_{x+1}\Delta u_x.$$

Hence operating by Σ on both sides we get
$$u_xv_x = \Sigma(u_x\Delta v_x) + \Sigma(v_{x+1}\Delta u_x),$$
i.e.,
$$\Sigma(u_x\Delta v_x) = u_xv_x - \Sigma(v_{x+1}\Delta u_x).$$

Example. Find the value of Σxa^x.

We have
$$\Delta a^x = a^x(a-1).$$

Hence
$$\Sigma xa^x(a-1) = xa^x - \Sigma a^{x+1} = xa^x - \frac{a^{x+1}}{a-1} + C$$
$$= xa^x - \frac{a^{x+1}-a}{a-1},$$
so that
$$\Sigma xa^x = \frac{xa^x}{a-1} - \frac{a^{x+1}-a}{(a-1)^2}.$$

2.9. Miscellaneous Forms. When the nth term of the series proposed for integration does not fall under any of the given forms, it is often possible to conjecture the form of the sum by inspection from a general knowledge of the effects of the operation Δ on it. Thus if $\phi(x)$ be a rational function of x, then $\psi(x)$, defined by the relation $\Delta a^x\phi(x) = a^x\psi(x)$, will also be a rational function.

Example. Sum when possible the series

$$\frac{1^2.x}{2.3}+\frac{2^2.x^2}{3.4}+\frac{3^2.x^3}{4.5}+\text{.............to } n \text{ terms.}$$

Here the nth term $=\dfrac{n^2.x^n}{(n+1)(n+2)}$,

and
$$S_n=\frac{n^2.x^n}{(n+1)(n+2)}+\Sigma\frac{n^2.x^n}{(n+1)(n+2)}.$$

As the summation has reference to n, we can legitimately assume

$$\Sigma\frac{n^2x^n}{(n+1)(n+2)}=\frac{an+b}{n+1}x^n.$$

Operating by Δ we get

$$\frac{n^2x^n}{(n+1)(n+2)}=x^n\left\{x.\frac{a(n+1)+b}{n+2}-\frac{an+b}{n+1}\right\}$$

$$=x^n.\ \frac{a(x-1)n^2+(2a+b)(x-1)n+(a+b)x-2b}{(n+1)(n+2)}.$$

Equating the coefficients on both sides, we must have

$$a(x-1)=1,\quad (2a+b)(x-1)=0,\quad (a+b)x-2b=0\ ;$$

whence
$$x=4,\quad a=\frac{1}{3},\quad b=-\frac{2}{3}.$$

Thus the proposed series can be summed if $x=4$, and then

$$S_n=\frac{n^2.4^n}{(n+1)(n+2)}+\frac{4^n}{3}\cdot\frac{n-2}{n+1}+C$$

$$=\frac{4^{n+1}}{3}\cdot\frac{n-1}{n+2}+C.$$

Putting $n=1$, we get $C=\dfrac{2}{3}$. Hence

$$S_n=\frac{4^{n+1}}{3}\cdot\frac{n-1}{n+2}+\frac{2}{3}.$$

2.10. Condition of integrability of the function

$$\tan^{-1}\frac{1}{p+qx+rx^2}.$$

The above function is not always integrable in finite terms, and here we try to find out the condition satisfied by p, q and r so that it may be integrable.

We know that

$$\Delta\tan^{-1}\frac{1}{ax+b}=\tan^{-1}\left\{\frac{-a}{1+ab+b^2+(a^2+2ab)x+a^2x^2}\right\}.$$

Assume $\Delta\tan^{-1}\dfrac{1}{ax+b}=\tan^{-1}\dfrac{1}{p+qx+rx^2}$.

Hence comparing the above expressions we have

$$1+ab+b^2=-ap,\quad a+2b=-q,\quad a=-r,$$

whence

$$b=\frac{r-q}{2},$$

so that we have

$$1-\frac{r.(r-q)}{2}+\left(\frac{r-q}{2}\right)^2=pr,$$

or

$$q^2-r^2=4\,(pr-1),$$

which is the condition to be satisfied that the given function may be integrable. Thus

$$\Sigma\tan^{-1}\frac{1}{p+qx+rx^2}=C-\tan^{-1}\frac{2}{2rx+q-r}. \tag{1}$$

Example 1. Sum the series

$$\tan^{-1}\frac{1}{1+1+1^2}+\tan^{-1}\frac{1}{1+2+2^2}+\ldots\ldots+\tan^{-1}\frac{1}{1+x+x^2}.$$

Here the coefficients satisfy the condition of integrability. Hence by (1)

$$S_x=\Sigma\tan^{-1}\frac{1}{1+(x+1)+(x+1)^2}=C-\tan^{-1}\frac{1}{2(x+1)}.$$

Putting $x=1$, we have $\tan^{-1}\dfrac{1}{1+1+1^2}=C-\tan^{-1}\dfrac{1}{2}$,

so that $C=\dfrac{\pi}{4}$.

Hence $$S_x = \frac{\pi}{4} - \tan^{-1}\frac{1}{x+1}. \quad S_\infty = \frac{\pi}{4}.$$

This example can be directly summed thus.

We have $$\Delta\tan^{-1}\frac{1}{x} = -\tan^{-1}\frac{1}{1+x+x^2}.$$

Hence $$\Sigma\tan^{-1}\frac{1}{1+x+x^2} = C - \tan^{-1}\frac{1}{x},$$

whence the result follows at once.

Example 2. Sum the series

$$\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{13} + \tan^{-1}\frac{1}{27} + \ldots\ldots + \tan^{-1}\frac{1}{\frac{1}{2}(3x+5x^2)}.$$

The condition of integrability is satisfied here. Hence by (1)

$$S_x = \Sigma\tan^{-1}\frac{1}{\frac{1}{2}\{3(x+1)+5(x+1)^2\}}$$

$$= C - \tan^{-1}\frac{2}{5x+4}.$$

Putting $x=1$, we have $\tan^{-1}\frac{1}{4} = C - \tan^{-1}\frac{2}{9}$, whence $C = \tan^{-1}\frac{1}{2}$.

Hence $S_x = \tan^{-1}\frac{1}{2} - \tan^{-1}\frac{2}{5x+4}$. $S_\infty = \tan^{-1}\frac{1}{2}$.

2.11. Another method of summing such series can be obtained thus:

Since $$\Delta\tan^{-1}u_x = \tan^{-1}\frac{\Delta u_x}{1+u_x u_{x+1}},$$

we have

(1) $$\Sigma\tan^{-1}\frac{\Delta u_x}{1+u_x u_{x+1}} = C + \tan^{-1}u_x.$$

Hence, if the proposed function can be put in form (1), it is easily evaluated.

Example 1. Evaluate $\Sigma \tan^{-1} \dfrac{n}{1+x(x-1)n^2}$.

Now
$$\Sigma \tan^{-1} \frac{n}{1+x(x-1)n^2} = \Sigma \tan^{-1} \frac{\Delta n(x-1)}{1+nx.n(x-1)}$$
$$= C + \tan^{-1} n(x-1).$$

Example 2. Sum the series whose xth term is

$$\tan^{-1} \frac{1}{10x^2-24x+12}.$$

Here
$$S_x = \Sigma \tan^{-1} \frac{1}{10(x+1)^2-24(x+1)+12}$$
$$= \Sigma \tan^{-1} \frac{1}{10x^2-4x-2} = \Sigma \tan^{-1} \frac{10}{100x^2-40x-20}$$
$$= \Sigma \tan^{-1} \frac{10}{1+100x^2-40x-21}$$
$$= \Sigma \tan^{-1} \frac{\Delta(10x-7)}{1+(10x-7)(10x+3)}.$$

Hence
$$S_x = C + \tan^{-1}(10x-7).$$

Putting $x=1$, we have $C = \tan^{-1}(-\frac{1}{2}) - \tan^{-1}(3) = \tan^{-1}(7)$.

Thus
$$S_x = \tan^{-1}(7) + \tan^{-1}(10x-7).$$

The same result can also be obtained from (1) of § 2.10.

2.12 Sum of the series

$$a_1.1^n t + a_2(1^n+2^n)t^2 + a_3(1^n+2^n+3^n)t^3 + \ldots\ldots$$
$$+ a_x(1^n+2^n+\ldots+x^n)t^x + \ldots\ldots$$

when the sum of the series

$$f(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_x t^x + \ldots$$

is known.

Here the general term $= a_x(1^n+2^n+\ldots+x^n)t^x$
$$= a_x t^x (E+E^2+\ldots+E^x).0^n$$
$$= a_x t^x \frac{E^{x+1}-E}{E-1}.0^n$$

(the operators having reference to 0)

$$=a_x t^x \frac{E}{\Delta}(E^x-1).0^n$$

$$=\frac{E}{\Delta}\left\{a_x t^x E^x - a_x t^x\right\}0^n.$$

Hence the required sum

$$=\frac{E}{\Delta}\left\{f(tE)-f(t)\right\}.0^n.$$

Example 1. Sum the series

$$1^n+\frac{1^n+2^n}{2!}+\frac{1^n+2^n+3^n}{3!}+\dots+\frac{1^n+2^n+\dots+x^n}{x!}+\dots$$

Here the general term $=\frac{1^n+2^n+\dots+x^n}{x!}$

$$=\frac{1}{x!}\,\frac{E}{\Delta}(E^x-1).0^n$$

(the operators having reference to 0)

$$=\frac{E}{\Delta}\left\{\frac{E^x}{x!}-\frac{1}{x!}\right\}0^n.$$

Hence the required sum

$$=\frac{E}{\Delta}\left\{\left(E+\frac{E^2}{2!}+\dots+\frac{E^x}{x!}+\dots\right)-\left(1+\frac{1}{2!}+\frac{1}{3!}+\dots\right)\right\}0^n$$

$$=\frac{E}{\Delta}\left\{e^E-e\right\}0^n=e.\frac{E}{\Delta}\left(e^\Delta-1\right)0^n$$

$$=e(1+\Delta)\left\{1+\frac{\Delta}{2!}+\frac{\Delta^2}{3!}+\dots+\frac{\Delta^{n-1}}{n!}+\frac{\Delta^n}{(n+1)!}\right\}0^n$$

$$=e\left\{0^n+\frac{3}{2!}\Delta 0^n+\frac{4}{3!}\Delta^2 0^n+\dots+\frac{n+2}{(n+1)!}\Delta^n 0^n\right\}.$$

Example 2. Sum the series

$$1^n.t+(1^n+2^n)t^2+(1^n+2^n+3^n)t^3+\dots$$

Here the general term $=(1^n+2^n+3^n+\dots+x^n)t^x$

$$=t^x.\frac{E}{\Delta}(E^x-1)0^n=\frac{E}{\Delta}(t^xE^x-t^x).0^n.$$

Hence the required sum

$$=\frac{E}{\Delta}\left\{(1+tE+t^2E^2+\dots)-(1+t+t^2+\dots)\right\}0^n$$

$$=\frac{E}{\Delta}\left\{(1-tE)^{-1}-(1-t)^{-1}\right\}0^n$$

$$=\frac{tE}{(1-tE)(1-t)}.0^n=\frac{t}{1-t}.\frac{E}{1-tE}0^n$$

$$=\frac{t}{(1-t)^2}.\left\{1+\frac{\Delta}{1-tE}\right\}0^n$$

$$=\frac{t}{(1-t)^2}\left\{1+\frac{\Delta}{1-t}\left(1-\frac{t\Delta}{1-t}\right)^{-1}\right\}0^n$$

$$=\frac{1}{(1-t)^2}\left\{t.0^n+\frac{t}{1-t}\Delta 0^n+\frac{t^2}{(1-t)^2}\Delta^2 0^n+\dots+\frac{t^n}{(1-t)^n}\Delta^n 0^n\right\}.$$

MISCELLANEOUS EXAMPLES

Example 1. If Σ'_n denote the sum of the first n terms of the series $u_0, u_2, u_4, \dots$, shew that

$$\Sigma'_n u_x=\frac{1}{2}\left\{\Sigma-\frac{1}{2}+\frac{\Delta}{4}-\frac{\Delta^2}{8}+\dots\right\}(u_{2n}-u_0). \qquad \text{(Boole.)}$$

We have $\Sigma'_n u_x=u_0+u_2+u_4+\dots+u_{2n-2}$.

$$\begin{aligned}\therefore\quad 2\,\Sigma'_n u_x&=u_0+u_1+u_2+\dots+u_{2n-2}+u_{2n-1}\\&\quad+u_0-u_1+u_2-\dots+u_{2n-2}-u_{2n-1}\\&=(\Sigma u_{2n}-\Sigma u_0)+(u_0-u_1+u_2-\dots-u_{2n-1}).\end{aligned}$$

Now $u_0-u_1+u_2-\dots+u_{2n-2}-u_{2n-1}$

$$=\frac{E^{2n}-1}{-E-1}u_0=-\frac{1}{2}\left(1+\frac{\Delta}{2}\right)^{-1}(u_{2n}-u_0)$$

$$=-\frac{1}{2}\left(1-\frac{\Delta}{2}+\frac{\Delta^2}{4}-\dots\right)(u_{2n}-u_0).$$

Hence $\Sigma'_n u_x = \frac{1}{2}\left\{\Sigma - \frac{1}{2} + \frac{\Delta}{4} - \frac{\Delta^2}{8} - \dots\right\}(u_{2n} - u_0)$.

Example 2. If $\Delta u_x = u_{x+\omega} - u_x$ and $\lambda = \frac{a^\omega}{1-a^\omega}$, shew that

$$u_x + \lambda\Delta u_x + \lambda^2\Delta^2 u_x + \dots + \lambda^n\Delta^n u_x$$
$$= a^{-x}\left\{(a^\omega - 1)\Sigma a^x u_x + \lambda^n \Sigma a^{x+\omega}\Delta^{n+1}u_x\right\}.$$

[*C. U., M.A. & M.Sc. 1936.*]

We have

$$u_x + \lambda\Delta u_x + \lambda^2\Delta^2 u_x + \dots + \lambda^n\Delta^n u_x = \frac{1-\lambda^{n+1}\Delta^{n+1}}{1-\lambda\Delta}u_x$$
$$= (1-\lambda\Delta)^{-1}u_x - \lambda^{n+1}(1-\lambda\Delta)^{-1}\Delta^{n+1}u_x$$
$$= (a^\omega - 1)(a^\omega E - 1)^{-1}u_x + \lambda^n a^\omega (a^\omega E - 1)^{-1}\Delta^{n+1}u_x$$
$$= a^{-x}(a^\omega - 1)(E-1)^{-1}a^x u_x + \lambda^n a^{-x}a^\omega(E-1)^{-1}a^x\Delta^{n+1}u_x$$
$$= a^{-x}\left\{(a^\omega - 1)\Sigma a^x u_x + \lambda^n\Sigma a^{x+\omega}\Delta^{n+1}u_x\right\}.$$

Example 3. Sum to x terms the series

$$\frac{1}{\sin\theta\sin 2\theta} + \frac{1}{\sin 2\theta\sin 3\theta} + \frac{1}{\sin 3\theta\sin 4\theta} + \dots\dots$$

Here $$S_x = \Sigma u_{x+1} = \Sigma\frac{1}{\sin(x+1)\theta\sin(x+2)\theta}.$$

Now $$\Delta\cot x\theta = -\frac{\sin\theta}{\sin x\theta\sin(x+1)\theta}.$$

$$\therefore\quad \Delta\cot(x+1)\theta = \frac{-\sin\theta}{\sin(x+1)\theta\sin(x+2)\theta}.$$

Hence $$S_x = C - \frac{\cot(x+1)\theta}{\sin\theta}.$$

Putting $x=1$, we have $C = \frac{1}{\sin\theta\sin 2\theta} + \frac{\cot 2\theta}{\sin\theta} = \frac{\cot\theta}{\sin\theta}$.

Thus $$S_x = \frac{\cot\theta - \cot(x+1)\theta}{\sin\theta}.$$

Example 4. Sum to x terms and to infinity the series

$$\tan\theta+\frac{1}{2}\tan\frac{\theta}{2}+\frac{1}{4}\tan\frac{\theta}{4}+\ldots$$

$$S_x=\Sigma u_{x+1}=\Sigma\frac{1}{2^x}\tan\frac{\theta}{2^x}.$$

Now $$\Delta\frac{1}{2^x\tan\dfrac{\theta}{2^x}}=\frac{1}{2^{x+1}}\tan\frac{\theta}{2^{x+1}}.$$

$\therefore$ $$\Sigma\frac{1}{2^x}\tan\frac{\theta}{2^x}=\frac{1}{2^{x-1}\tan\dfrac{\theta}{2^{x-1}}}+C.$$

Putting $x=1$, we get $C=-2\cot 2\theta$.

Hence $$\Sigma\frac{1}{2^x}\tan\frac{\theta}{2^x}=\frac{1}{2^{x-1}\tan\dfrac{\theta}{2^{x-1}}}-2\cot 2\theta.$$

Sum to infinity is $$\frac{1}{\theta}-2\cot 2\theta.$$

Example 5. Sum to x terms and to infinity the series

$$\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{38}+2^2\tan^{-1}\frac{1}{268}+\ldots$$

The series can be put thus

$$\tan^{-1}\frac{1}{4+3}+2\tan^{-1}\frac{1}{4.8+3.2}+2^2\tan^{-1}\frac{1}{4.8^2+3.2^2}+\ldots$$

Hence $$u_{x+1}=2^x\tan^{-1}\frac{1}{4.8^x+3.2^x}$$

and $$S_x=\Sigma u_{x+1}=\Sigma 2^x\tan^{-1}\frac{1}{4.8^x+3.2^x}.$$

Now $$\Delta 2^x\tan^{-1}\frac{1}{2^x}=2^x\tan^{-1}\frac{1}{4.8^x+3.2^x}.$$

Therefore $$S_x=C+2^x\tan^{-1}\frac{1}{2^x}.$$

Putting $x=1$, we have $C=\tan^{-1}\frac{1}{7}-2\tan^{-1}\frac{1}{2}=-\frac{\pi}{4}$

$$\therefore\quad S_x=2^x\tan^{-1}\frac{1}{2^x}-\frac{\pi}{4}.\qquad S_\infty=1-\frac{\pi}{4}.$$

Example 6. Sum the series

$$\tan^{-1}\frac{1}{4n^3+3n}+2\tan^{-1}\frac{1}{32n^3+6n}+4\tan^{-1}\frac{1}{256n^3+12n}+\ldots$$

Here $\qquad u_{x+1}=2^x\tan^{-1}\dfrac{1}{4.2^{3x}n^3+3.2^x.n}.$

Now $\qquad \tan^{-1}\dfrac{1}{n}+\tan^{-1}\dfrac{1}{4n^3+3n}=2\tan^{-1}\dfrac{1}{2n}.$

$$\therefore\quad \tan^{-1}\frac{1}{4n^3+3n}=2\tan^{-1}\frac{1}{2n}-\tan^{-1}\frac{1}{n}.$$

Hence $\qquad S_x=\Sigma u_{x+1}=\Sigma 2^x\tan^{-1}\dfrac{1}{4.(2^xn)^3+3.2^x.n}$

$$=\Sigma 2^x\left\{2\tan^{-1}\frac{1}{2.2^xn}-\tan^{-1}\frac{1}{2^xn}\right\}$$

$$=\Sigma\left\{2^{x+1}\tan^{-1}\frac{1}{2^{x+1}n}-2^x\tan^{-1}\frac{1}{2^xn}\right\}$$

$$=2^x\tan^{-1}\frac{1}{2^xn}+C.$$

Putting $x=1$, we get $C=-\tan^{-1}\frac{1}{n}$.

Thus $S_x=2^x\tan^{-1}\dfrac{1}{2^xn}-\tan^{-1}\dfrac{1}{n}.\qquad S_\infty=\dfrac{1}{n}-\tan^{-1}\dfrac{1}{n}.$

Example 7. Sum the series

$$\tan^{-1}\frac{1}{2f^0(n)}-\tan^{-1}\frac{1}{2f^1(n)}+\tan^{-1}\frac{1}{2f^2(n)}-\ldots$$

where $f^0(n)=n,\quad f^1(n)=4n^3+3n,\quad f^2(n)=4\{f(n)\}^3+3f(n),$ etc.

Here $$u_{x+1}=(-1)^x \tan^{-1}\frac{1}{2f^x(n)}.$$

Now we have $$\tan^{-1}\frac{1}{n}+\tan^{-1}\frac{1}{4n^3+3n}=2\tan^{-1}\frac{1}{2n}.$$

Hence putting $f^x(n)$ for n and multiplying throughout by $(-1)^x$, we get

$$(-1)^x \tan^{-1}\frac{1}{f^x(n)}+(-1)^x\tan^{-1}\frac{1}{f^{x+1}(n)}=2(-1)^x\tan^{-1}\frac{1}{2f^x(n)},$$

or $$\Delta(-1)^{x+1}\tan^{-1}\frac{1}{f^x(n)}=2(-1)^x\tan^{-1}\frac{1}{2f^x(n)}=2u_{x+1},$$

whence $$\Sigma u_{x+1}=\frac{1}{2}(-1)^{x+1}\tan^{-1}\frac{1}{f^x(n)}+C.$$

Putting $x=1$, we get $C=\frac{1}{2}\tan^{-1}\frac{1}{n}$.

Thus $$S_x=\frac{1}{2}\left\{\tan^{-1}\frac{1}{n}+(-1)^{x+1}\tan^{-1}\frac{1}{f^x(n)}\right\}.$$

$$S_\infty=\frac{1}{2}\tan^{-1}\frac{1}{n}.$$

EXAMPLES

1. Sum to x terms the series

(i) $1.3.5.7+3.5.7.9+5.7.9.11+\dots$

(ii) $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\dots$

(iii) $\frac{1}{1.2.3.4}+\frac{3}{2.3.4.5}+\frac{5}{3.4.5.6}+\dots$

(iv) $\frac{1}{1.3.5.7}+\frac{1}{3.5.7.9}+\frac{1}{5.7.9.11}+\dots$

(v) $\frac{2}{1.3.4}+\frac{3}{2.4.5}+\frac{4}{3.5.6}+\dots$

(*vi*) $\frac{10}{1.3.5}+\frac{12}{3.5.7}+\frac{14}{5.7.9}+\ldots$

(*vii*) $1.2.4+2.3.5+3.4.6+\ldots$

(*viii*) $\frac{3}{1.4.10}+\frac{5}{4.7.13}+\frac{7}{7.10.16}+\ldots$

(*ix*) $\frac{1}{2.6.10}+\frac{1}{4.8.12}+\frac{1}{6.10.14}+\ldots$

(*x*) $\frac{3}{2.5.8}+\frac{5}{5.8.11}+\frac{7}{8.11.14}+\ldots$

(*xi*) $1.3.5\cos\theta+3.5.7\cos 2\theta+5.7.9\cos 3\theta+\ldots$

(*xii*) $1+2a\cos\theta+3a^2\cos 2\theta+4a^3\cos 3\theta+\ldots$

2. Sum to x terms the series

(*i*) $1^2.2+2^2.2^2+3^2.2^3+4^2.2^4+\ldots$

(*ii*) $1.3^2+3.5^2+5.7^2+\ldots$

(*iii*) $\frac{1^2.9}{3.5}+\frac{2^2.9^2}{5.7}+\frac{3^2.9^3}{7.9}+\ldots$

(*iv*) $\frac{2}{1.3.5}-\frac{5.11}{3.5.7}+\frac{8.11^2}{5.7.9}-\ldots$

3. Sum to x terms and to infinity the series

(i) $\frac{3}{5.7}-\frac{9}{7.29}+\frac{27}{29.79}-\frac{81}{79.245}+\ldots$

(ii) $\frac{3}{1.2}.\frac{1}{2}+\frac{4}{2.3}.\frac{1}{2^2}+\frac{5}{3.4}.\frac{1}{2^3}+\ldots$

(iii) $\frac{5}{1.2.3}.\frac{1}{2}+\frac{6}{2.3.4}.\frac{1}{2^2}+\frac{7}{3.4.5}.\frac{1}{2^3}+\ldots$

(iv) $\frac{1}{\sqrt{2}(1+\sqrt{2})}+\frac{1}{(1+\sqrt{2})(2+\sqrt{2})}+\frac{1}{(2+\sqrt{2})(3+\sqrt{2})}+\ldots$

(v) $\frac{1.3}{2}+\frac{3.5}{2^2}+\frac{5.7}{2^3}+\ldots$

4. Sum to x terms and to infinity the series where u_x is equal to

(*i*) $\tan^{-1}\dfrac{1}{2x^2}$. (*ii*) $\tan^{-1}\dfrac{1}{34x^2-8x-8}$.

(*iii*) $\tan^{-1}\dfrac{1}{74x^2-12x-18}$. (*iv*) $\tan^{-1}\dfrac{1}{26x^2-16x-4}$.

5. Sum to x terms and to infinity the series

(*i*) $(\tan\theta)^2+\dfrac{1}{2}\left(\tan\dfrac{\theta}{2}\right)^2+\dfrac{1}{4}\left(\tan\dfrac{\theta}{4}\right)^2+\dots$

(*ii*) $\tan\theta(\sec\theta)^2+\dfrac{1}{2}\tan\dfrac{\theta}{2}\left(\dfrac{1}{2}\sec\dfrac{\theta}{2}\right)^2$

$+\dfrac{1}{4}\tan\dfrac{\theta}{4}\left(\dfrac{1}{4}\sec\dfrac{\theta}{4}\right)^2+\dots$

(*iii*) $\sin\theta\left(\sin\dfrac{\theta}{2}\right)^2+2\sin\dfrac{\theta}{2}\left(\sin\dfrac{\theta}{4}\right)^2+4\sin\dfrac{\theta}{4}\left(\sin\dfrac{\theta}{8}\right)^2+\dots$

(*iv*) $\dfrac{1}{\left(2\cos\dfrac{\theta}{2}\right)^2}+\dfrac{1}{\left(4\cos\dfrac{\theta}{4}\right)^2}+\dfrac{1}{\left(8\cos\dfrac{\theta}{8}\right)^2}+\dots$

(*v*) $(\sin\theta)^4+4\left(\sin\dfrac{\theta}{2}\right)^4+4^2\left(\sin\dfrac{\theta}{4}\right)^4+\dots$

6. Sum to x terms the series

(i) $\dfrac{a-1}{a+1}+2.\dfrac{a^2-1}{a^2+1}+4.\dfrac{a^4-1}{a^4+1}+\dots$

(*ii*) $\dfrac{a}{a^2-1}+\dfrac{a^2}{a^4-1}+\dfrac{a^4}{a^8-1}+\dots$

7. Sum to x terms the series

(*i*) $\sin\theta+\sin 3\theta+\sin 5\theta+\dots$

(*ii*) $\cos\theta+\cos 3\theta+\cos 5\theta+\dots$

(*iii*) $\dfrac{1}{\sin\theta}+\dfrac{1}{\sin 2\theta}+\dfrac{1}{\sin 4\theta}+\dots$

(*iv*) $\dfrac{1}{\cos\theta\cos 2\theta}+\dfrac{1}{\cos 2\theta\cos 3\theta}+\dfrac{1}{\cos 3\theta\cos 4\theta}+\dots$

(*v*) $\dfrac{1}{\sin\theta\ \cos 2\theta}-\dfrac{1}{\cos 2\theta\ \sin 3\theta}+\dfrac{1}{\sin 3\theta\cos 4\theta}-\dots$

(*vi*) $\dfrac{1}{\cos\theta\ \sin 2\theta}-\dfrac{1}{\sin 2\theta\cos 3\theta}+\dfrac{1}{\cos 3\theta\sin 4\theta}-\dots$

(*vii*) $\dfrac{1}{\sin\theta\sin 3\theta}-\dfrac{1}{\sin 2\theta\sin 4\theta}+\dfrac{1}{\sin 3\theta\sin 5\theta}-\dots$

(*viii*) $\left(\dfrac{1}{\cos\theta}\right)^2+\left(\dfrac{2}{\cos 2\theta}\right)^2+\left(\dfrac{4}{\cos 4\theta}\right)^2+\left(\dfrac{8}{\cos 8\theta}\right)^2+\dots$

8. Sum to x terms the series where u_x is equal to

(*i*) x^3+7x. (*ii*) $\dfrac{2x+3}{x(x+1)}3^{-x}$.

(*iii*) $\dfrac{(x+1)^2}{x(x+2)(x+3)(x+4)}$. (*iv*) $2^x.x.\dfrac{x\,!}{(2x+1)\,!}$.

9. Find the sum to x terms of the series

(*i*) $2.2+7.4+14.8+23.16+34.32+\dots$

(*ii*) $1^2+\dfrac{3^2}{2}+\dfrac{5^2}{2^2}+\dfrac{7^2}{2^3}+\dots$

10. Expand $\Sigma\phi(x)\cos ax$ in a series of differences of $\phi(x)$.

11. Shew that

$$u_0+u_1\cos 2\theta+u_2\cos 4\theta+\dots=\frac{u_0}{2}-\frac{\Delta u_0}{4\sin^2\theta}+\frac{\Delta^2 u_0}{8\sin^3\theta}\sin\theta$$

$$+\frac{\Delta^3 u_0}{16\sin^4\theta}\cos 2\theta-\frac{\Delta^4 u_0}{32\sin^5\theta}\sin 3\theta+\dots$$

12. If $\phi(x)=v_0+v_1x+v_2x^2+v_3x^3+\dots.$ shew that

$$u_0v_0+u_1v_1x+u_2v_2x^2+\dots=u_0\phi(x)+x\phi'(x).\Delta u_0+\frac{x^2}{2\,!}\phi''(x).\Delta^2u_0+\dots;$$

and if $\phi(x)=v_0+v_1x+v_2x^{(2)}+v_3x^{(3)}+\dots$ shew that

$$u_0v_0+u_1v_1x+u_2v_2x^{(2)}+\dots$$
$$=u_0\phi(x)+x\Delta\phi(x).\Delta u_0+\frac{x^{(2)}}{2!}\Delta^2\phi(x).\Delta^2u_0+\dots$$

[Guderman, *Crelle*, VII, 306.]

13. Sum to infinity the series

$$0^r+1^r.\frac{z}{x}+2^r.\frac{z(z-1)}{x(x-1)}+3^r.\frac{z(z-1)(z-2)}{x(x-1)(x-2)}+\dots$$

14. If $1^n+2^n+3^n+\dots+m^n\equiv S_n$ and $m(m+1)=p$, shew that $S_n=p^2f(p)$ or $(2m+1)pf(p)$ according as n is odd or even, $f(p)$ being a polynomial.

[*Nouvelles Annales*, X, 199.]

15. Shew that

$$\frac{1}{z^2}+\frac{1}{(z+1)^2}+\frac{1}{(z+2)^2}+\dots=\frac{1}{z}+\frac{1}{2z(z+1)}+\frac{1.2}{3.z(z+1)(z+2)}+\dots$$

16. If $\phi(x)\equiv v_0+v_1x+v_2x^2+\dots$, shew that

$$a_ru_rx^r+a_{r+n}u_{r+n}x^{r+n}+a_{r+2n}u_{r+2n}x^{r+2n}+\dots$$
$$=\frac{1}{n}\left\{\Sigma\{\alpha^{n-r}\phi(\alpha x)\}u_0+\Sigma\{\alpha^{n-r+1}\phi'(\alpha x)\}\Delta u_0.x+\dots\right\}$$

where α is an nth root of unity.

17. Prove that

$$\Sigma u_nx^n=\frac{u_0-x^nu_n}{1-x}+\frac{x}{(1-x)^2}(\Delta u_0-x^n\Delta u_n)$$
$$+\frac{x^2}{(1-x)^3}(\Delta^2u_0-x^n\Delta^2u_n)+\dots$$

18. Prove that

$$\Delta(x!)=x(x!)$$

and hence sum to n terms the series

$$1+3.2!+7.3!+13.4!+21.5!+\dots$$

CHAPTER III

APPROXIMATE SUMMATION OF SERIES: BERNOULLI'S AND EULER'S NUMBERS

3.1. In the previous chapter several methods have been given for finite summation. But when such finite expression cannot be obtained, we can find out an approximate expression for the sum. In many cases the difference between the actual value and the approximate value so obtained will be very small.

3.2. Euler-Maclaurin Sum Formula. *Expression for* Σu *in terms of the differential coefficients of* u_x.

We have $\quad \Sigma u_x = \Delta^{-1} u_x + C = (e^D - 1)^{-1} u_x + C.$

Now

$$\frac{t}{e^t - 1} = \frac{t}{t + \dfrac{t^2}{2!} + \dfrac{t^3}{3!} + \dots} = 1 - \frac{t}{2} + \frac{t^2}{12} - \frac{t^4}{720} + \dots \tag{1}$$

Here $-\dfrac{t}{2}$ is the only term involving odd power of t. For

we have $\quad \dfrac{t}{e^t - 1} + \dfrac{t}{2} = \dfrac{t}{2} . \dfrac{e^t + 1}{e^t - 1} = 1 + \dfrac{t^2}{12} - \dfrac{t^4}{720} + \dots$

and these expressions remain unchanged when t is changed into $-t$, so that we are sure that the right-hand expression contains on expension only even powers of t.

It is usual to express (1) in the form

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \frac{B_1}{2!} t^2 - \frac{B_3}{4!} t^4 + \frac{B_5}{6!} t^6 - \dots$$
$$+ (-1)^{n+1} \frac{B_{2n-1}}{(2n)!} t^{2n} + \dots \tag{2}$$

The quantities B_1, B_3, B_5, etc., are called Bernoulli's numbers.* Their values are

$$B_1 = \frac{1}{6},\ B_3 = \frac{1}{30},\ B_5 = \frac{1}{42},\ B_7 = \frac{1}{30},\ B_9 = \frac{5}{66}, \dots$$

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* James Bernoulli—*Ars Conjectandi.*

These numbers begin to increase after B_5 and these ultimately form a divergent series.

Dividing both sides of (2) by t, we have

$$(3)\qquad (e^t-1)^{-1}=\frac{1}{t}-\frac{1}{2}+\frac{B_1}{2!}t-\frac{B_3}{4!}t^3+\frac{B_5}{6!}t^5-\dots$$

Hence

$$(4)\quad \Sigma u_x=C+\int u_x dx-\frac{1}{2}u_x+\frac{B_1}{2!}\frac{du_x}{dx}-\frac{B_3}{4!}\frac{d^3u_x}{dx^3}+\frac{B_5}{6!}\frac{d^5u_x}{dx^5}-\dots$$

$$=C+\int u_x dx-\frac{1}{2}u_x+\frac{1}{12}\frac{du_x}{dx}-\frac{1}{720}\frac{d^3u_x}{dx^3}+\frac{1}{30240}\frac{d^5u_x}{dx^5}-\dots$$

The value of the constant C is to be determined from the particular conditions of the problem.

If u_x be a rational integral function of x, the integral is at once obtained.

This theorem will be very useful when finite summation is impracticable.

3.3. Bernoulli's Numbers. The numerical values of Bernoulli's numbers may be obtained thus:

Let $f(t)=\frac{t}{e^t-1}$, then by Maclaurin's theorem

$$(1)\qquad f(t)=f(0)+\left(\frac{df}{dt}\right)_0 . t+\left(\frac{d^2f}{dt^2}\right)_0 . \frac{t^2}{2!}+\left(\frac{d^3f}{dt^3}\right)_0 . \frac{t^3}{3!}+\dots$$

But by §3.2(2)

$$f(t)=1-\frac{t}{2}+B_1\frac{t^2}{2!}-B_3\frac{t^4}{4!}+B_5\frac{t^6}{6!}-\dots$$

so that all the odd differential coefficients in (1) except the first vanish, *i.e.*, $\left(\frac{d^{2n+1}f}{dt^{2n+1}}\right)_0=0$, for $n=1, 2, 3\dots$

Now $(e^t-1)f(t)=t$, and differentiating n times, we have

$$(2) \qquad e^t\left(1+\frac{d}{dt}\right)^n f(t)-\frac{d^n}{dt^n}f(t)=0, \text{ when } n>1,$$
$$=1, \text{ when } n=1.$$

Putting $t=0$ and $n=1, 2, 3, \ldots,$ in (2) we have

$n=1$, $f(0)=1$,

$n=2$, $1+2\frac{df}{dt}=0$, or $\left(\frac{df}{dt}\right)_0=-\frac{1}{2}$,

$n=3$, $1+3\frac{df}{dt}+3\frac{d^2f}{dt^2}=0$, or $\left(\frac{d^2f}{dt^2}\right)_0=\frac{1}{6}=B_1$,

$n=5$, $1+5\frac{df}{dt}+10\frac{d^2f}{dt^2}+10\frac{d^3f}{dt^3}+5\frac{d^4f}{dt^4}=0$,

$$i.e., \; 1-\frac{5}{2}+\frac{10}{6}+5\frac{d^4f}{dt^4}=0, \text{ or } \left(\frac{d^4f}{dt^4}\right)_0=-\frac{1}{30}.$$

Proceeding in this manner we get

$$B_1=\frac{1}{6},\ B_3=\frac{1}{30},\ B_5=\frac{1}{42},\ B_7=\frac{1}{30},\ \ldots$$

Example 1. Find the value of $\Sigma x^2=1^2+2^2+\ldots+(x-1)^2$.

We have $\Sigma x^2=C+\int x^2dx-\frac{1}{2}x^2+\frac{1}{12}\frac{d}{dx}x^2$

$$=C+\frac{x^3}{3}-\frac{x^2}{2}+\frac{x}{6}.$$

Putting $x=2$, we get $C=0$. Hence

$$\Sigma x^2=\frac{x(x-1)(2x-1)}{6}.$$

We have obtained here a finite expression for the sum.

Example 2. Sum the series $\Sigma \frac{1}{x^2}$.

We have by §3.2(4)

$$\Sigma\frac{1}{x^2}=C-\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{12}\cdot\frac{-2}{x^3}-\frac{1}{720}\cdot\frac{-2.3.4}{x^5}-\dots$$

$$=C-\frac{1}{x}-\frac{1}{2x^2}-\frac{1}{6x^3}+\frac{1}{30x^5}-\dots$$

To determine the value of C put $x=\infty$, then

$$\sum_1^\infty\frac{1}{x^2}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots \textit{ ad infinitum}=C.$$

But it is known that

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots \textit{ ad infinitum}=\frac{\pi^2}{6},$$

so that $C=\frac{\pi^2}{6}$. Hence

$$\Sigma\frac{1}{x^2}=\frac{\pi^2}{6}-\frac{1}{x}-\frac{1}{2x^2}-\frac{1}{6x^3}+\frac{1}{30x^5}-\dots$$

If x be large, we need only take a few terms on the right-hand side.

3.4. When the sum to infinity is unknown or infinite, we can determine the value of C approximately by giving to x some such value which renders terms after the first few terms on the right side of §3.2(4) practically negligible.

Thus taking the series $1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{x}$

we have $$S_x=\frac{1}{x}+\Sigma\frac{1}{x}$$

$$=\frac{1}{x}+C+\log x-\frac{1}{2x}-\frac{1}{12x^2}+\frac{1}{120x^4}-\dots$$

$$=C+\log x+\frac{1}{2x}-\frac{1}{12x^2}+\frac{1}{120x^4}-\dots$$

To determine C, put $x=10$, then

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{10}=C+\log 10+\frac{1}{20}-\frac{1}{1200}+\frac{1}{1200000}-\ldots$$

Here we can neglect higher terms as their values are infinitesimally small. Simplifying and giving to $\log 10$ its value $2{\cdot}302585$, we get $C={\cdot}577215$ approximately. Hence

$$S_x={\cdot}577215+\log x+\frac{1}{2x}-\frac{1}{12x^2}+\frac{1}{120x^4}-\cdots$$

3.5. Stirling's Theorem.* **Approximate value of** $1.2.3\ldots x$.

$$\text{Put } u_x=1.2.3\ldots x$$

then
$$\log u_x=\log 1+\log 2+\log 3+\ldots+\log x$$
$$=\log x+\Sigma \log x.$$

Hence by §3.2(4) we have

$$\log u_x=\log x+C+\int\log x\,dx-\frac{1}{2}\log x$$
$$+\frac{B_1}{2!}\frac{d}{dx}\log x-\frac{B_3}{4!}\frac{d^3}{dx^3}\log x+\ldots$$

(1) $$=C+\left(x+\frac{1}{2}\right)\log x-x+\frac{1}{12x}-\frac{1}{360x^3}+\ldots$$

To determine C, suppose x to be infinitely large, then

(2) $$\log(1.2.3\ldots x)=C+(x+\tfrac{1}{2})\log x-x.$$

(3) Therefore $$1.2.3\ldots x=e^{C-x}.x^{x+\frac{1}{2}}.$$

(4) Hence $$1.2.3\ldots 2x=e^{C-2x}.(2x)^{2x+\frac{1}{2}}.$$

Again multiplying (3) by 2^x we get

(5) $$2.4.6\ldots 2x=2^x e^{C-x}.x^{x+\frac{1}{2}}.$$

Dividing (4) by (5) we get

(6) $$1.3.5.7\ldots(2x-1)=2^{x+\frac{1}{2}}e^{-x}.x^x.$$

Again dividing (5) by (6) we have

(7) $$\frac{2.4.6\ldots 2x}{1.3.5\ldots(2x-1)}=\frac{e^C\sqrt{x}}{\sqrt{2}}.$$

---

* **Stirling**—*Methodus Differentialis* (1730).

But by Wallis' theorem,[*] since x is infinitely large, we have

$$\frac{2.4.6 \ldots (2x-2)\sqrt{(2x)}}{1.3.5 \ldots (2x-1)} = \frac{\sqrt{\pi}}{\sqrt{2}}.$$

Hence from (7) we have

$$\sqrt{(2x)}\sqrt{\pi} = e^{C}\sqrt{x}$$

or

$$e^{C} = \sqrt{(2\pi)}.$$

$$\therefore \quad C = \log \sqrt{2\pi}.$$

Hence

$$\log u_x = \log \sqrt{(2\pi)} + \log x^{x+\frac{1}{2}} - x + \frac{1}{12x} - \frac{1}{360x^3} + \ldots$$

or

$$(8) \qquad 1.2.3 \ldots\ldots x = \sqrt{(2\pi x)}\; x^x e^{-x + \frac{1}{12x} - \frac{1}{360x^3} + \ldots}$$

$$(9) \qquad = \sqrt{(2\pi x)}\left(\frac{x}{e}\right)^x\left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} + \ldots\right).$$

For infinitely large values of x we have[†]

$$(10) \qquad 1.2.3 \ldots x = \sqrt{(2\pi x)}\left(\frac{x}{e}\right)^x.$$

Example. Find an approximate value of

$$\frac{3.5 \ldots (2x+1)}{2.4 \ldots 2x},$$

when x is large but not infinite.

As x is large but does not tend to infinity, assume

$$1.2.3 \ldots x = \sqrt{(2\pi x)}\left(\frac{x}{e}\right)^x\left(1 + \frac{1}{12x}\right)$$

neglecting the higher powers of $\frac{1}{x}$.

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[*] **Wallis**—*Arithmetica Infinitorum* (1656).

[†] An elementary proof of (10) was given by **Glaisher** in *Quarterly Journal of Mathematics*, 1878. See also **Raabe**, *Crelles Journal*, XXV.

Then $$1.2.3\ldots 2x=\sqrt{(2\pi.2x)}.\left(\frac{2x}{e}\right)^{2x}\left(1+\frac{1}{24x}\right)$$

and $$2.4.6\ldots 2x=2^{x}\sqrt{(2\pi x)}.\left(\frac{x}{e}\right)^{x}\left(1+\frac{1}{12x}\right)$$

so that $$3.5.7\ldots(2x-1)=\sqrt{2}\left(\frac{2x}{e}\right)^{x}\left(1-\frac{1}{24x}\right).$$

$$\therefore\ \frac{3.5.7\ldots(2x-1)}{2.4.6\ldots 2x}=\frac{1}{\sqrt{(\pi x)}}\left(1-\frac{1}{8x}\right).$$

Hence $$\frac{3.5.7\ldots(2x-1)(2x+1)}{2.4.6\ldots\ldots 2x}=\frac{2x+1}{\sqrt{(\pi x)}}\left(1-\frac{1}{8x}\right)$$

$$=\frac{1}{\sqrt{\pi}}\left(2\sqrt{x}+\frac{3}{4\sqrt{x}}\right).$$

3.6. The chief use of Euler-Maclaurin Sum formula §3.2(4) is to convert a series into another which converges more rapidly than the given series. The series so formed is convergent for the first few terms and then diverges, but by keeping only the convergent portion we obtain an approximate value of the given series. It may be necessary in some cases to find the sum of the first few terms in ordinary manner before we apply this formula. This gives a very close approximation to the value of the function.

As an illustration take *Ex.* 2 of §3.3. We have

$$\Sigma\frac{1}{x^{2}}=C-\frac{1}{x}-\frac{1}{2x^{2}}-\frac{B_{1}}{x^{3}}+\frac{B_{3}}{x^{5}}-\frac{B_{5}}{x^{7}}+\ldots$$

Taking the series between the limits ∞ and 1, we have

$$1+\frac{1}{2^{2}}+\frac{1}{3^{2}}+\ldots\ \textit{ad infinitum}=1+\frac{1}{2}+B_{1}-B_{3}+B_{5}-\ldots$$

As the Bernoulli's numbers are convergent up to B_5 and then diverge, let us stop at B_5. Then

$$1+\tfrac{1}{2}+B_1-B_3+B_5=1{\cdot}65714,$$

giving a first approximation. Again retaining the first few terms and then applying the formula we have

$$\Sigma\frac{1}{x^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\sum_{x=5}^{\infty}\frac{1}{x^5}$$

$$=\frac{205}{144}+\frac{1}{5}+\frac{1}{2{,}5^2}+\frac{B_1}{5^3}-\frac{B_3}{5^5}+\ldots$$

Here the convergence is so rapid that if we stop at B_1 we have

$$\Sigma\frac{1}{x^2}=\frac{205}{144}+\frac{1}{5}+\frac{1}{2{\cdot}5^2}+\frac{B_1}{5^3}=1{\cdot}64494,$$

which differs very slightly from the true value 1·64493. To get a similar approximation from the original series, thousands of terms would have to be taken.

3.7. If the terms of the series be alternately positive and negative, we can make use of the formula for $\Sigma(-1)^x\phi(x)$ to convert it into a rapidly convergent one. Thus taking the series

$$\phi(0)-\phi(1)+\phi(2)-\ldots\ldots \textit{ ad infinitum}$$

we have this equivalent to

$$(1-E+E^2-\ldots\ldots)\phi(0)$$

$$=(1+E)^{-1}\phi(0)=\frac{1}{2}\left(1+\frac{\Delta}{2}\right)^{-1}\phi(0),$$

so that

$$\text{(1)}\qquad \Sigma(-1)^x\phi(x)=\frac{1}{2}\left\{1-\frac{\Delta}{2}+\frac{\Delta^2}{4}-\frac{\Delta^3}{8}+\ldots\right\}\phi(0).$$

The formula (1) transforms the series into a rapidly convergent one.

For example, take the series

$$S=\frac{1}{x(x+1)(x+2)}-\frac{1}{(x+1)(x+2)(x+3)}+\ldots\ldots$$

Applying (1) we have

$$S=\frac{1}{2}\left\{1-\frac{\Delta}{2}+\frac{\Delta^2}{4}-\dots\right\}\frac{1}{x(x+1)\,(x+2)}$$

$$=\frac{1}{2}\left\{\frac{1}{x(x+1)\,(x+2)}+\frac{1}{2}\cdot\frac{3}{x(x+1)\,(x+2)\,(x+3)}\right.$$

$$\left.+\frac{1}{4}\cdot\frac{3.4}{x(x+1)\,(x+2)\,(x+3)\,(x+4)}+\dots\right\}$$

which converges more rapidly than original series S.

In particular, putting $x=4$, we have

$$\frac{1}{4.5.6}-\frac{1}{5.6.7}+\frac{1}{6.7.8}-\dots$$

$$=\frac{1}{2}\left\{\frac{1}{4.5.6}+\frac{1}{2}\cdot\frac{3}{4.5.6.7}+\frac{1}{4}\cdot\frac{3.4}{4.5.6.7.8}+\dots\right\}$$

for rapid convergence.

3.8. Expression for $\Sigma a^x\phi(x)$ in terms of differential coefficients of $\phi(x)$.

We have $\Sigma a^x\phi(x)=(E-1)^{-1}a^x\phi(x)=a^x(aE-1)^{-1}\phi(x)$

$$=a^x(ae^D-1)^{-1}\phi(x).$$

Now by Herschel's theorem, we have

$$(ae^D-1)^{-1}=(aE-1)^{-1}e^{0\cdot D}=\frac{1}{a-1}\left\{1+\frac{a\Delta}{a-1}\right\}^{-1}e^{0\cdot D},$$

where Δ operates on 0 only.

$$\therefore\quad \Sigma a^x\phi(x)=\frac{a^x}{a-1}\left\{1+\frac{a\Delta}{a-1}\right\}^{-1}e^{0\cdot D}\phi(x)$$

(1) $$=\frac{a^x}{a-1}\left\{1+\frac{a\Delta}{a-1}\right\}^{-1}\left\{\phi(x)+0.D\phi(x)+\frac{0^2}{2!}.D^2\phi(x)+\dots\right\};$$

Putting $a=-1$, we have

$$\Sigma(-1)^x\phi(x)=(E-1)^{-1}(-1)^x\phi(x)$$

$$=(-1)^x(-e^D-1)^{-1}\phi(x)$$

$$=(-1)^{x-1}(e^D+1)^{-1}\phi(x)$$

$$=(-1)^{x-1}\{(e^{D}-1)^{-1}-2\,(e^{2D}-1)^{-1}\}\phi(x)$$

$$=(-1)^{x-1}\left\{\frac{1}{D}-\frac{1}{2}+\frac{B_1}{2!}D-\frac{B_3}{4!}D^3+\dots\right.$$

$$\left.-2\left(\frac{1}{2D}-\frac{1}{2}+\frac{B_1}{2!}2D-\frac{B_3}{4!}2^3D^3+\dots\right)\right\}\phi(x).$$

Thus

$$(2)\qquad \Sigma(-1)^x\phi(x)=(-1)^{x-1}\left\{\frac{1}{2}-\frac{B_1}{2!}(2^2-1)D\right.$$

$$\left.+\frac{B_3}{4!}(2^4-1)D^3-\dots\right\}\phi(x),$$

where B_1, B_3, etc., are Bernoulli's numbers.

3.9. Expression for $\Sigma^n u_x$ in terms of differential coefficients of u_x.

Since $\Sigma^n=(e^D-1)^{-n}$

we shall expand $(e^t-1)^{-n}$ in ascending powers of t.

Put $v_n=(e^t-1)^{-n}$;

then
$$\frac{dv_n}{dt}=\frac{-ne^t}{(e^t-1)^{n+1}}=\frac{-n(e^t-1)-n}{(e^t-1)^{n+1}}$$

$$=-nv_n-nv_{n+1}.$$

Hence
$$nv_{n+1}=-\left(\frac{d}{dt}+n\right)v_n.$$

Similarly
$$(n-1)v_n=-\left(\frac{d}{dt}+n-1\right)v_{n-1},$$

$$(n-2)v_{n-1}=-\left(\frac{d}{dt}+n-2\right)v_{n-2},$$

....

$$1.v_2=-\left(\frac{d}{dt}+1\right)v_1.$$

Multiplying these together, we have

$$(n-1)!\,v_n=(-1)^{n-1}\left(\frac{d}{dt}+n-1\right)\left(\frac{d}{dt}+n-2\right)\dots\left(\frac{d}{dt}+1\right)(e^t-1)^{-1}.$$

$$\therefore \quad (e^t-1)^{-n}=\frac{(-1)^{n-1}}{(n-1)!}\left(\frac{d}{dt}+n-1\right)\left(\frac{d}{dt}+n-2\right)\ldots\left(\frac{d}{dt}+1\right)$$

$$\times\left\{\frac{1}{t}-\frac{1}{2}+\frac{B_1}{2!}t-\frac{B_3}{4!}t^3+\frac{B_5}{6!}t^5-\ldots\right\}. \tag{2}$$

Putting D for t in (2) we get the expression for Σ^n in terms of Bernoulli's numbers.

Example. Find an expression for $\Sigma^3 u_x$ in terms of D.

We have from (2)

$$(e^t-1)^{-3}=\frac{1}{2}\left(\frac{d}{dt}+2\right)\left(\frac{d}{dt}+1\right)\left(\frac{1}{t}-\frac{1}{2}+\frac{B_1}{2!}t-\frac{B_3}{4!}t^3+\ldots\right)$$

$$=\frac{1}{2}\left(\frac{d^2}{dt^2}+3\frac{d}{dt}+2\right)\left(\frac{1}{t}-\frac{1}{2}+\frac{B_1}{2!}t-\frac{B_3}{4!}t^3+\ldots\right)$$

$$=\frac{1}{2}\left\{\frac{2}{t^3}-\frac{3}{t^2}+\frac{2}{t}-1+\frac{3B_1}{2!}+B_1t-\frac{3.2B_3t}{4!}-\frac{3.3.B_3}{4!}t^2+\ldots\right\}$$

$$=\frac{1}{t^3}-\frac{3}{2t^2}+\frac{1}{t}-\frac{3}{8}+\frac{19}{240}t-\frac{1}{160}t^2+\ldots\ldots$$

Hence

$$\Sigma^3u_x=\iiint u_x(dx)^3-\frac{3}{2}\iint u_x(dx)^2+\int u_xdx-\frac{3}{8}u_x$$

$$+\frac{19}{240}\frac{du_x}{dx}-\frac{1}{160}\frac{d^2u_x}{dx^2}+\ldots$$

3.10. Expressions for Bernoulli's Numbers B_{2n-1}.

We have from §3.2(2)

$$\frac{t}{e^t-1}=1-\frac{1}{2}t+\frac{B_1}{2!}t^2-\frac{B_3}{4!}t^4+\frac{B_5}{6!}t^6-\ldots\ldots \tag{1}$$

$$+(-1)^{n+1}\frac{B_{2n-1}}{(2n)!}t^{2n}+\ldots$$

Again by Herschel's theorem

$$\frac{t}{e^t-1}=\frac{\log e^t}{e^t-1}=\frac{\log E}{E-1}e^{0\cdot t}=\frac{\log(1+\Delta)}{\Delta}e^{0\cdot t}. \tag{2}$$

Hence comparing the coefficients of t^{2n} in (1) and (2), we have

$$(-1)^{n+1}\frac{B_{2n-1}}{(2n)!}=\frac{\log(1+\Delta)}{\Delta}\cdot\frac{0^{2n}}{(2n)!}.$$

$$(3)\qquad \therefore\ B_{2n-1}=(-1)^{n+1}\left\{1-\frac{\Delta}{2}+\frac{\Delta^2}{3}-\ldots+\frac{\Delta^{2n}}{2n+1}\right\}0^{2n}.$$

As the series (1) does not include terms involving t^{2n+1} $(n\geq 1)$, its coefficients are zero. Hence

$$(4)\qquad \left\{1-\frac{\Delta}{2}+\frac{\Delta^2}{3}-\ldots+\frac{\Delta^{2n}}{2n+1}-\frac{\Delta^{2n+1}}{2n+2}\right\}0^{2n+1}=0.$$

Again since $\Delta^m 0^n=m(\Delta^{m-1}0^{n-1}+\Delta^m 0^{n-1})$, by §1.20,
we have $\Delta^m 0^{2n+1}=m(\Delta^{m-1}0^{2n}+\Delta^m 0^{2n})$.

Hence
$$\Delta^{m-1}0^{2n}=\frac{\Delta^m 0^{2n+1}}{m}-\Delta^m 0^{2n}.$$

Giving to m the values 1, 2, 3, ... and substituting in (3) we have

$$B_{2n-1}=(-1)^{n+1}\left\{\left(\Delta-\frac{\Delta^2}{2^2}+\frac{\Delta^3}{3^2}-\ldots+\frac{\Delta^{2n+1}}{(2n+1)^2}\right)0^{2n+1}\right.$$
$$\left.-\left(\Delta-\frac{\Delta^2}{2}+\frac{\Delta^3}{3}-\ldots+\frac{\Delta^{2n+1}}{2n+1}\right)0^{2n}\right\}.$$

But since
$$\left(\Delta-\frac{\Delta^2}{2}+\frac{\Delta^3}{3}-\ldots\right)0^{2n}$$
$$=\log(1+\Delta)0^{2n}=D0^{2n}=0,$$

we have

$$(5)\qquad B_{2n-1}=(-1)^{n+1}\left\{\Delta-\frac{\Delta^2}{2^2}+\frac{\Delta^3}{3^2}-\ldots+\frac{\Delta^{2n+1}}{(2n+1)^2}\right\}0^{2n+1}.$$

3.11. Another elegant expression for B_{2n-1} is obtained from the expansion of $\frac{1}{1+e^t}$. We have

$$\frac{1}{1+e^t}=\frac{1}{1+E}\cdot e^{0\cdot t}$$
$$=\frac{1}{2+\Delta}\left\{1+0.t+\frac{0^2.t^2}{2!}+\ldots\ldots+\frac{0^{2n-1}.t^{2n-1}}{(2n-1)!}+\ldots\right\}.$$

Again

$$\frac{1}{1+e^t}=\frac{1}{2}-\frac{B_1}{2!}(2^2-1)t+\frac{B_3}{4!}(2^4-1)t^3-\ldots\ldots$$

$$+(-1)^n\frac{B_{2n-1}}{(2n)!}(2^{2n}-1)t^{2n-1}+\ldots\ldots$$

Hence, comparing the coefficients of t^{2n-1}, we get

$$(-1)^n\frac{B_{2n-1}}{(2n)!}(2^{2n}-1)=\frac{1}{2+\Delta}\cdot\frac{0^{2n-1}}{(2n-1)!}$$

or

(6) $$B_{2n-1}=(-1)^n\frac{2n}{2^{2n}-1}\cdot\frac{1}{2+\Delta}0^{2n-1},$$

i.e.,

$$B_{2n-1}=(-1)^n\frac{n}{2^{2n}-1}\left\{1-\frac{\Delta}{2}+\frac{\Delta^2}{2^2}-\frac{\Delta^3}{2^3}+\ldots-\frac{\Delta^{2n-1}}{2^{2n-1}}\right\}0^{2n-1}.$$

3.12. Expressions for tan θ, cot θ, etc., in terms of B_{2n-1}. We have from §3.2

$$\frac{t}{2}\cdot\frac{e^t+1}{e^t-1}=1+\frac{B_1}{2!}t^2-\frac{B_3}{4!}t^4+\frac{B_5}{6!}t^6-\ldots\ldots$$

Putting $t=2\theta$ in it, we have

(1) $$\theta\coth\theta=1+\frac{B_1}{2!}2^2\theta^2-\frac{B_3}{4!}2^4\theta^4+\frac{B_5}{6!}2^6\theta^6-\ldots\ldots$$

$$\ldots+(-1)^{n-1}\frac{B_{2n-1}}{(2n)!}2^{2n}\theta^{2n}-\ldots\ldots$$

Again, putting $t=2i\theta$, we get

(2) $$\theta\cot\theta=1-\frac{B_1}{2!}2^2\theta^2-\frac{B_3}{4!}2^4\theta^4-\frac{B_5}{6!}2^6\theta^6-\ldots\ldots$$

$$\ldots-\frac{B_{2n-1}}{(2n)!}2^{2n}\theta^{2n}-\ldots\ldots$$

Expressions for $\tan\theta$ and $\operatorname{cosec}\theta$ are obtained from the identities

$$\tan\theta = \cot\theta - 2\cot 2\theta$$

$$\operatorname{cosec}\theta = \cot\frac{\theta}{2} - \cot\theta.$$

Thus

$$(3)\quad \tan\theta = \frac{2^2(2^2-1)}{2!}B_1\theta + \frac{2^4(2^4-1)}{4!}B_3\theta^3 + \frac{2^6(2^6-1)}{6!}B_5\theta^5 + \\ \ldots\ldots + \frac{2^{2n}(2^{2n}-1)}{(2n)!}B_{2n-1}\theta^{2n-1} + \ldots$$

and

$$(4)\quad \operatorname{cosec}\theta = \frac{1}{\theta} + 2(2-1)\frac{B_1\theta}{2!} + 2(2^3-1)\frac{B_3}{4!}\theta^3 + 2(2^5-1)\frac{B_5}{6!}\theta^5 + \\ \ldots\ldots + \frac{2(2^{2n-1}-1)}{(2n)!}B_{2n-1}\theta^{2n-1} + \ldots$$

Similarly

$$(5)\quad \tanh\theta = \frac{2^2(2^2-1)}{2!}B_1\theta - \frac{2^4(2^4-1)}{4!}B_3\theta^3 + \frac{2^6(2^6-1)}{6!}B_5\theta^5 - \\ \ldots\ldots + \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)}{(2n)!}B_{2n-1}\theta^{2n-1} + \ldots$$

$$(6)\quad \operatorname{cosech}\theta = \frac{1}{\theta} - \frac{2(2-1)}{2!}B_1\theta + \frac{2(2^3-1)}{4!}B_3\theta^3 - \frac{2(2^5-1)}{6!}B_5\theta^5 - \\ \ldots\ldots + (-1)^n\frac{2(2^{2n-1}-1)}{(2n)!}B_{2n-1}\theta^{2n-1} + \ldots$$

3.13. Again since $\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\ldots\ldots$

we have $\log\sin\theta = \log\theta + \log\left(1 - \frac{\theta^2}{\pi^2}\right) + \log\left(1 - \frac{\theta^2}{2^2\pi^2}\right) + \ldots\ldots$

Hence $$\cot\theta=\frac{1}{\theta}-\frac{2\theta}{\pi^2}\left(1-\frac{\theta^2}{\pi^2}\right)^{-1}-\frac{2\theta}{2^2\pi^2}\left(1-\frac{\theta^2}{2^2\pi^2}\right)^{-1}-\dots$$

or

(1) $$\theta\cot\theta=1-\frac{2\theta^2}{\pi^2}\left(1-\frac{\theta^2}{\pi^2}\right)^{-1}-\frac{2\theta^2}{2^2\pi^2}\left(1-\frac{\theta^2}{2^2\pi^2}\right)^{-1}-\dots$$

$$=1-\frac{2\theta^2}{\pi^2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\dots\right)$$

$$-\frac{2\theta^4}{\pi^4}\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\dots\right)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$-\frac{2\theta^{2n}}{\pi^{2n}}\left\{1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\dots\right\}+\dots$$

Comparing the coefficient of θ^{2n} in (1) with §3.12(2) we have

$$\frac{B_{2n-1}2^{2n}}{(2n)!}=\frac{2}{\pi^{2n}}\left\{1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\dots\right\}$$

(2) *i.e.*, $$B_{2n-1}=\frac{2\,(2n)!}{(2\pi)^{2n}}\left\{1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\dots\dots\right\}.$$

Let $$S_{2n}=1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\dots\dots$$

Then $$\frac{1}{2^{2n}}S_{2n}=\frac{1}{2^{2n}}+\frac{1}{4^{2n}}+\frac{1}{6^{2n}}+\dots\dots$$

Hence $$\left(1-\frac{1}{2^{2n}}\right)S_{2n}=1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\dots\dots$$

where all the terms whose denominators are multiples of 2 have been removed.

The expression for (2) then becomes

(3) $$B_{2n-1}=\frac{2\,(2n)!}{\left(1-\frac{1}{2^{2n}}\right)(2\pi)^{2n}}\left\{1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\dots\dots\right\}.$$

If we take the next prime number 3 and multiply

$$\left(1-\frac{1}{2^{2n}}\right)S_{2n} \text{ by } 1-\frac{1}{3^{2n}},$$

we shall deprive the series of all terms involving multiples of 3, and so on. Thus we get

$$\left(1-\frac{1}{2^{2n}}\right)\left(1-\frac{1}{3^{2n}}\right)\left(1-\frac{1}{5^{2n}}\right)\ldots\left(1-\frac{1}{p^{2n}}\right)S_{2n}=1+\frac{1}{q^{2n}}+\ldots$$

where 2, 3, $5,\ldots,p$ are the succession of natural primes up to p, and q is the prime next to p. Taking q to be very large, $\frac{1}{q^{2n}}$ becomes very small. Hence

$$S_{2n}=\left\{\left(1-\frac{1}{2^{2n}}\right)\left(1-\frac{1}{3^{2n}}\right)\left(1-\frac{1}{5^{2n}}\right)\ldots\ldots\right\}^{-1}$$

where the succession of primes continues to infinity. Hence

$$B_{2n-1}=\frac{2(2n)!}{(2\pi)^{2n}}\cdot\frac{1}{\left(1-\frac{1}{2^{2n}}\right)\left(1-\frac{1}{3^{2n}}\right)\left(1-\frac{1}{5^{2n}}\right)\ldots\ldots}.$$

For large values of n this formula will be of great use.

3.14. We have from Algebra

$$\frac{1}{2n-1}>\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\ldots\ldots>\frac{1}{(2n-1)2^{2n-1}}.$$

Hence from §3.13(2) we have

$$\frac{B_{2n+1}}{B_{2n-1}}=\frac{(2n+2)(2n+1)S_{2n+2}}{(2\pi)^2S_{2n}}$$

$$>\frac{(2n+2)(2n+1)\left\{1+\frac{1}{(2n+1)2^{2n+1}}\right\}}{(2\pi)^2\left\{1+\frac{1}{2n-1}\right\}}$$

$$>\frac{(2n)^2-1}{4\pi^2}.$$

Hence $B_{2n+1} > B_{2n-1}$ provided $4n^2-1 > 4\pi^2$ or $n > \{\pi^2+\frac{1}{4}\}^{\frac{1}{2}}$, *i.e.*, if $n > 3{\cdot}16$. Now $B_5 < B_7$. Hence

$$B_5 < B_7 < B_9 < \ldots\ldots$$

Again since S_{2n} tends to 1 for $n \longrightarrow \infty$, we have B_{2n-1} tending to ∞ for $n \longrightarrow \infty$. Thus

Bernoulli's numbers are all positive and they increase after B_5 *and* $\lim\limits_{n\longrightarrow\infty} B_{2n-1} \longrightarrow \infty$.

The value of $\frac{B_{2n-1}}{(2n)!}$ ultimately approaches to equality with a geometrical series having for its common ratio $\frac{1}{4\pi^2}$.

3.15. Again from §3.12(2) we have

$$\theta\cos\theta = \sin\theta\left\{1 - \frac{B_1}{2!}2^2\theta^2 - \frac{B_3}{4!}2^4\theta^4 - \ldots - \frac{B_{2n-1}}{(2n)!}2^{2n}\theta^{2n} - \ldots\ldots\right\}.$$

Expanding $\cos\theta$ and $\sin\theta$ and equating the coefficients of θ^{2n+1} we have

$$\frac{(-1)^n}{(2n)!} = \frac{(-1)^n}{(2n+1)!} - \frac{(-1)^{n-1}}{(2n-1)!}\,\frac{B_1}{2!}2^2 + \ldots + \frac{1}{3!}\,\frac{B_{2n-3}}{(2n-2)!}2^{2n-2} - \frac{B_{2n-1}}{(2n)!}2^{2n}.$$

This is a recurrence formula connecting the Bernoulli's numbers.

3.16. Euler's Numbers. Another series of numbers closely analogous to those of Bernoulli was given by Euler.* They appear in the coefficients in the expansion of $\sec\theta$. Thus

$$(1)\qquad \sec\theta = 1 + \frac{E_2}{2!}\theta^2 + \frac{E_4}{4!}\theta^4 + \ldots + \frac{E_{2n}}{(2n)!}\theta^{2n} + \ldots$$

Again we have

$$\sec\theta = 4\Sigma(-1)^{n-1}\,\frac{(2n-1)\pi}{(2n-1)^2\pi^2 - 4\theta^2}$$

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* *Inst. Cal. Diff.*, §224.

(2) $$=4\Big[\frac{1}{\pi}\Big\{1-\Big(\frac{2\theta}{\pi}\Big)^2\Big\}^{-1}-\frac{1}{3\pi}\Big\{1-\Big(\frac{2\theta}{3\pi}\Big)^2\Big\}^{-1}+\dots\dots$$
$$+(-1)^{n-1}\frac{1}{(2n-1)\pi}\Big\{1-\Big(\frac{2\theta}{(2n-1)\pi}\Big)^2\Big\}^{-1}-\dots\Big],$$

where $\theta\neq\frac{1}{2}\pi,\ \frac{3}{2}\pi,\ \frac{5}{2}\pi,\dots\dots\dots$

Hence the coefficient of θ^{2n} in (2) is

$$4.\frac{2^{2n}}{\pi^{2n+1}}\Big\{1-\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}-\dots+(-1)^{n-1}\frac{1}{(2n-1)^{2n+1}}-\dots\Big\}.$$

Comparing the above with the coefficient of θ^{2n} in (1) we have

(3) $$E_{2n}=\frac{2.2^{2n+1}(2n)!}{\pi^{2n+1}}\Big\{1-\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}-\dots\dots$$
$$+(-1)^{n-1}\frac{1}{(2n-1)^{2n+1}}-\dots\Big\}.$$

This can easily be transformed into the form *

$$E_{2n}=\frac{2.2^{2n+1}(2n)!}{\pi^{2n+1}}\cdot\frac{1}{\Big(1+\frac{1}{3^{2n+1}}\Big)\Big(1-\frac{1}{5^{2n+1}}\Big)\Big(1+\frac{1}{7^{2n+1}}\Big)\dots}$$

3.16. Recurrence Formula for Euler's Numbers.

Since $$\sec\theta=1+\Sigma\frac{E_{2n}\theta^{2n}}{(2n)!}\ .$$

we have $$1=\cos\theta\Big\{1+\Sigma\frac{E_{2n}\theta^{2n}}{(2n)!}\Big\},$$

or $$1=\Big\{1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\dots+(-1)^n\frac{\theta^{2n}}{(2n)!}+\dots\dots\dots\Big\}$$
$$\times\Big\{1+\frac{E_2}{2!}\theta^2+\frac{E_4}{4!}\theta^4+\dots+\frac{E_{2n}}{(2n)!}\theta^{2n}+\dots\dots\Big\}.$$

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* **Euler**—*Introd. in Anal. Inf.*, § 284.

Hence
$$\frac{E_{2n}}{(2n)!} - \frac{E_{2n-2}}{2!(2n-2)!} + \frac{E_{2n-4}}{4!(2n-4)!} - \dots\dots + (-1)^{n-1}\frac{E_2}{(2n-2)!2!} + (-1)^n\frac{1}{(2n)!} = 0,$$
or

(1) $$E_{2n} = {}^{2n}C_2E_{2n-2} - {}^{2n}C_4E_{2n-4} + \dots + (-1)^{n-2}\,{}^{2n}C_2E_2 + (-1)^{n-1}.$$

This equation enables us to calculate the values of E's. Thus
$$E_2 = 1,\ E_4 = 5,\ E_6 = 61,\ E_8 = 1385,\ E_{10} = 50521, \text{ etc.}$$
Again since
$$1 > 1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \dots > 1 - \frac{1}{3^{2n+1}},$$
we have
$$\frac{E_{2n+2}}{E_{2n}} = \frac{(2n+2)\ (2n+1)4}{\pi^2}\frac{\left\{1 - \frac{1}{3^{2n+3}} + \frac{1}{5^{2n+3}} - \dots\right\}}{\left\{1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \dots\right\}}$$
$$> \frac{(2n+2)(2n+1)}{\pi^2}4\left\{(1 - \frac{1}{3^{2n+3}}\right\},$$
which constantly increases as n increases and is greater than 1 for $n=1$. Therefore $E_2 < E_4 < E_6 \dots$

Also $$\text{Lim}\left\{1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \dots\right\} = 1, \text{ when } n = \infty$$

and $$\lim_{n \to \infty} \frac{(2n)!\ 2^{2n+1}}{\pi^{2n+1}} = \infty. \qquad \text{Hence } \lim_{n \to \infty} E_{2n} = \infty.$$

Thus *Euler's numbers are all positive and continually increase in magnitude and have infinity for their upper limit.*

3.17. Factorial Coefficients. The coefficients of powers of x in the expansion of $x^{(n)}$ are usually termed as *Factorial Coefficients.* The numerical value of the coefficient of x^k is usually represented by C_k^n. Thus

(1) $$x^{(n)} = C_n^n x^n - C_{n-1}^n x^{n-1} + C_{n-2}^n x^{n-2} - \dots\dots + (-1)^{n-k}C_k^n x^k - \dots\dots + (-1)^{n-1}C_1^n x.$$

By Maclaurin's theorem we have

$$x^{(n)}=0^{(n)}+xD\,0^{(n)}+\frac{x^2D^20^{(n)}}{2!}+\ldots\ldots+\frac{x^kD^k0^{(n)}}{k!}+\ldots\ldots$$

so that

$$(2)\qquad (-1)^{n-k}C_k^n=\frac{D^k0^{(n)}}{k!},$$

the actual evaluation of which involves laborious calculations.

We can however obtain a recurrence formula, from which the values of C^{n+1} are easily obtained from those of C^n. For, we have

$$x^{(n+1)}=(x-n)x^{(n)},$$

so that

$$(3)\qquad C_k^{n+1}=C_{k-1}^n+nC_k^n.$$

The values of C_k^1 are 1, 0, 0,...., so that those of C_k^2 are 1, 1, 0, 0,...., for $k=1, 2, 3,...$

3.18. Again representing by C_k^{-n} the numerical value of the coefficient of $\frac{1}{x^k}$ in the expansion of $x^{(-n)}$ in negative powers of x, we have

$$x^{(-n)}=\frac{1}{x^n}-\frac{C_{n+1}^{-n}}{x^{n+1}}+\frac{C_{n+2}^{-n}}{x^{n+2}}-\ldots+(-1)^{k-n}\frac{C_k^{-n}}{x^k}+\ldots\ldots$$

Now $\Delta^{n-1}\frac{1}{x}=(-1)^{n-1}(n-1)!\,x^{(-n)}$,

hence
$$x^{(-n)}=\frac{(-1)^{n-1}}{(n-1)!}\Delta^{n-1}\frac{1}{x}=\frac{(-1)^{n-1}}{(n-1)!}\left\{\Delta^{n-1}\frac{1}{x+y}\right\}_{y=0}$$

(Δ operating on y only)

$$=\frac{(-1)^{n-1}}{(n-1)!}\left\{\Delta^{n-1}\left(\frac{1}{x}-\frac{y}{x^2}+\frac{y^2}{x^3}-\ldots\right)\right\}_{y=0}$$

$$=\frac{(-1)^{n-1}}{(n-1)!}\left\{-\frac{\Delta^{n-1}0}{x^2}+\frac{\Delta^{n-1}0^2}{x^3}-\ldots\right\}.$$

$$(1)\qquad \therefore\quad C_k^{-n}=\frac{\Delta^{n-1}0^{k-1}}{(n-1)!}.$$

Thus $x^{(-n)}=\Sigma(-1)^{k-n}\dfrac{\Delta^{n-1}0^{k-1}}{(n-1)\,!}\dfrac{1}{x^k}$, $k=n, n+1\ldots\ldots$,

Again since $\Delta^{n-1}0^{k-1}=(n-1)\{\Delta^{n-2}0^{k-2}+\Delta^{n-1}0^{k-2}\}$,

we have
$$C_k^{-n}=\left\{\frac{\Delta^{n-2}0^{k-2}}{(n-2)\,!}+\frac{\Delta^{n-1}0^{k-2}}{(n-2)\,!}\right\}$$

(2)
$$=C_{k-1}^{-(n-1)}+(n-1)C_{k-1}^{-n},$$

a recurrence formula from which the values of C_k are obtained from those of C_{k-1}.

3.19. Again since

$$x^n=0^n+x\Delta 0^n+\frac{x(x-1)}{2\,!}\Delta^2 0^n+\ldots$$

we have by §3.18(1)

(1)
$$x^n=\left\{xC_{n+1}^{-2}+x(x-1)C_{n+1}^{-3}+\ldots\right\}.$$

Similarly from § 1.19 we have

$$\frac{1}{x^n}=\frac{D^{n-1}0^{(n-1)}}{(n-1)\,!}x^{(-n)}-\frac{D^{n-1}0^{(n)}}{(n-1)\,!}x^{(-n-1)}+\ldots$$

$$=C_{n-1}^{n-1}x^{(-n)}-C_{n-1}^{n}x^{(-n-1)}+\ldots,\text{ by §3.17(2)}.$$

EXAMPLES

1. Find an approximate expression for the series

$$1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\ldots+\frac{1}{x^{2n}}.$$

2. Find an approximate expression for $\Sigma\dfrac{1}{x^5}$ and also the value of

$$1+\frac{1}{2^5}+\frac{1}{3^5}+\ldots\ldots \textit{ ad infinitum}$$

to ten places of decimals.

3. Find an expression for

$$1+\frac{1}{5^2}+\frac{1}{9^2}+\ldots\ldots\text{to } x \text{ terms},$$

and obtain an approximate value for the sum *ad infinitum*.

4. Show how you would convert the series

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\ldots\ldots$$

to a more rapidly converging series, and thus evaluate its sum correct to 5 places of decimals.

[*C U., M.A. & M.Sc., 1938*]

5. Transform the series $\frac{1}{l2}-\frac{1}{l3}+\frac{1}{l4}-\ldots$ into a rapidly convergent one.

6. If $u_0+u_1x+u_2x^2+\ldots=f(x)$, shew that

$$u_0v_x+u_1v_{x+1}+u_2v_{x+2}+\ldots=f(1)v_x+f'(1)\Delta v_x+\frac{f''(1)}{2!}\Delta^2v_x+\ldots$$

and apply this theorem to transform the series

$$x^{(-m)}+2\frac{(x+1)^{(-m)}}{2}+3\frac{(x+2)^{(-m)}}{2^2}+\ldots$$

to one proceeding by factorials only.

7. Find the sum to n terms and to infinity of the series

$$1+\frac{x^{(1)}}{z^{(1)}}+\frac{x^{(2)}}{z^{(2)}}+\frac{x^{(3)}}{z^{(3)}}+\ldots$$

[*C. U., M.A. & M.Sc., 1938*]

8. Shew that

$$1^n=\frac{1}{n+1}+\frac{1}{2}+\frac{n}{2!}B_1-\frac{n(n-1)(n-2)}{4!}B_3+\ldots$$

9. Prove that if n be an odd integer

$$\frac{1}{2}=nB_1-\frac{n(n-1)(n-2)}{3!}B_3+\frac{n(n-1)(n-2)(n-3)(n-4)}{5!}B_5-\ldots$$

to $(n-1)$ terms.

10. Shew that

$$B_{2n-1}=\left(\frac{1}{2}-\frac{\Delta}{3}+\frac{\Delta^2}{4}-\ldots\ldots\right)1^{2n-1}.$$

11. Shew that

$$\frac{1}{e^x-1}+\frac{1}{e^{2x}-1}+\frac{1}{e^{3x}-1}+\ldots\ldots \textit{ ad infinitum}$$

$$=\frac{C-\log x}{x}+\frac{1}{4}-\frac{B_1^2}{2.2!}x-\frac{B_3{}^2}{4.4!}x^3-\ldots\ldots$$

[*C. U., M.A. & M.Sc., 1938*]

and apply this to the summation of Lambert's series

$$\frac{x}{1-x}+\frac{x^2}{1-x^2}+\frac{x^3}{1-x^3}+\ldots,\text{ when } x \text{ is } \frac{1}{e} \text{ nearly.}$$

[*Zeitschrift*, VI, 407]

12. Expand $\Sigma\frac{1}{(ax+b)^n}$ in terms of successive differences of $\log(ax+b)$ and deduce

$$\Sigma \cot x=C+\left\{\log \sin x-\frac{\Delta}{2}\log \sin x+\frac{\Delta^3}{3}\log \sin x-\ldots\right\}.$$

13. Prove that the coefficient of θ^{2n} in the expansion of $\left(\frac{\theta}{\sin\theta}\right)^2$ is equal to $\frac{2^{2n}(2n-1)}{(2n)!}B_{2n-1}$.

14. Shew by Bernoulli's numbers or otherwise that

$$\frac{1^2}{1^2+1}\cdot\frac{2^2}{2^2+1}\cdot\frac{3^2}{3^2+1}\ldots \textit{ ad infinitum}=\frac{2\pi}{e^\pi-e^{-\pi}}.$$

15. Shew that the coefficient of $\frac{z^n}{n!}$ in

$$\int_0^z \log(1-e^{-t})dt-z\log z$$

is numericallye qual to $\frac{B_{n-2}}{n-1}$.

16. Find an expression for B_{2n-1} as a definite integral from the identity

$$\frac{e^x+1}{e^x-1}-\frac{2}{x}=4\int_0^\infty \frac{\sin xt}{e^{2\pi t}-1}\,dt.$$

17. Prove that

$$\frac{2^{2n-1}}{(4n)!\,2!}B_{4n-1}-\frac{2^{2n-3}}{(4n-4)!\,6!}B_{4n-5}+\ldots\ldots+\frac{(-1)^n n}{(4n+2)!}=0.$$

[*Grunert*, XXVII, 455]

18. Prove that

$$1-\frac{\pi^2}{2!}B_1-\frac{\pi^4}{4!}B_3-\ldots\ldots=0.$$

$$1-\frac{B_1}{2!}\frac{\pi^2}{2^2}-\frac{B_3}{4!}\frac{\pi^4}{2^4}-\ldots=\frac{\pi}{4}.$$

19. Shew that

$$1-\frac{1}{3^3}+\frac{1}{5^3}-\ldots=\frac{\pi^3}{32},$$

$$1-\frac{1}{3^5}+\frac{1}{5^5}-\ldots=\frac{5\pi^5}{1536}.$$

20. If $x\dfrac{d}{dx}\equiv D_1$, shew that

$$D_1^{n+1}f(x)\equiv xf'(x)+\frac{\Delta 1^n}{1}x^2f''(x)+\frac{\Delta^2 1^n}{2!}x^3f'''(x)+\ldots$$

21. Shew that if $k<n$, the coefficient of x^k in $\left(\dfrac{x}{e^x-1}\right)^n$

is

$$\frac{(-1)^k C^n_{n-k}}{(n-1)(n-2)\ldots(n-k)}.$$

[*Grunert*, XVIII, 315]

22. Show that $C^n_1-C^n_2+C^n_3-\ldots=0,$

and

$$B_1C^n_2-B_3C^n_4+B_5C^n_6-\ldots=\frac{(n-1)!}{2}\cdot\frac{n-1}{n+1}.$$

[*Grunert*, IX, 333]

23. Prove that

$$E_{2n}=2\int_0^\infty\frac{\theta^{2n}\,d\theta}{e^{\frac{\pi}{2}\theta}+e^{-\frac{\pi}{2}\theta}}$$

CHAPTER IV

INTERPOLATION AND MECHANICAL QUADRATURE

4.1. Interpolation. In mathematical and statistical researches it frequently happens that certain values of a function are found by experiment, observation or calculation and it is required to find other values of the function from them. The process of obtaining unknown values of the function, or the approximate form of the function, from the given values, is known as *Interpolation*.

Although differing fundamentally in the character of their respective functions, all mathematical tables are alike in giving the numerical values of the functions for certain assumed values of the argument so chosen that the intermediate values of the function may be readily derived by the process of interpolation.

As no series of particular values can determine a law, the problem of interpolation is an indeterminate one. The general form of the function is not known; only a series of isolated points on the curve representing the function is given. Between any two consecutive points we can draw an infinite number of curves and thus we may get an infinite number of curves passing through the given series of points. Consequently the analytical expressions of the curves are infinite in number. It becomes therefore necessary to make some assumption in order to have a problem with definite solution. The assumption usually made is that the function can be represented by a rational integral function of x of degree $n-1$, when n points on the curve or n other relations are known. Then the nth and all higher differences will vanish. It is with such functions that the actuary and the statistician are mainly concerned.

Extrapolation is sometimes defined as the process of obtaining *further* terms of a series as opposed to interpolation which is the process of finding *intermediate* terms.

4.2. Gregory-Newton Formula of Interpolation. When n consecutive equidistant values of the function $f(x)$ are known, the general expression of the function is obtained with the help of Gregory-Newton Formula. Let $f(a)$, $f(a+\omega)$, ... $f(a+\overline{n-1}\omega)$ be the given values, then by §1.9 we have

$$(1) \qquad f(a+x\omega)=f(a)+x\Delta f(a)+\frac{x(x-1)}{2!}\Delta^2 f(a)+\ldots\ldots$$

$$+\frac{x(x-1)(x-2)\ldots(x-n+2)}{(n-1)!}\Delta^{n-1}f(a).$$

As n entries are given, we can find the differences up to $\Delta^{n-1}f(a)$ from the table and $\Delta^n f(a)$ and higher differences all vanish by assumption, so that $f(a+x\omega)$ is a rational integral function of x of degree $n-1$. This represents an approximate expression of the function.

Taking $a=0$ and $\omega=1$, we have

$$(2) \qquad f(x)=f(0)+x\Delta f(0)+\frac{x(x-1)}{2!}\Delta^2 f(0)+\ldots\ldots$$

$$+\frac{x(x-1)\,(x-2)\ldots\ldots(x-n+2)}{(n-1)!}\Delta^{n-1}f(0).$$

Example 1. Values of x^4 are given for $x=2$, 4, 6, 8, 10 and 12; find the value of $(2{\cdot}8)^4$.

Completing the table we have

x	x^4	Δ	Δ^2	Δ^3	Δ^4	Δ^5
2	16					
		240				
4	256		800			
		1040		960		
6	1296		1760		384	
		2800		1344		0
8	4096		3104		384	
		5904		1728		
10	10000		4832			
		10736				
12	20736					

Here $a=2$, $\omega=2$, $x\omega=2{\cdot}8-2={\cdot}8$, so that $x={\cdot}4$;

$$\therefore\quad (2{\cdot}8)^4=16+{\cdot}4\times240+\frac{{\cdot}4({\cdot}4-1)}{2\,!}\times800+\frac{{\cdot}4({\cdot}4-1)({\cdot}4-2)}{3\,!}\times960$$

$$+\frac{{\cdot}4({\cdot}4-1)({\cdot}4-2)({\cdot}4-3)}{4\,!}\times384$$

$$=16+96-96+61{\cdot}44-15{\cdot}9744$$

$$=61{\cdot}4656.$$

This result is easily verified and found exact to the last figure.

Example 2. The Northampton 3 per cent. annuities for the ages 21, 25, 29, 33 and 37 are respectively 18·4708, 17·8144, 17·1070, 16·3432 and 15·5154. Find the annuity for the age 30.

		Δ	Δ^2	Δ^3	Δ^4
21	18·4708				
		−·6564.			
25	17·8144		−·0510		
		−·7074		−·0054	
29	17·1070		−·0564		−·0022
		−·7638		−·0076	
33	16·3432		−·0640		
		−·8278			
37	15·5154				

Here $a=21$, $\omega=4$ and $x\omega=9$, so that $x=2{\cdot}25$.

$$\therefore\quad f(30)=18{\cdot}4708-2{\cdot}25\times{\cdot}6564-\frac{2{\cdot}25\times1{\cdot}25}{2\,!}\times{\cdot}0510$$

$$-\frac{2{\cdot}25\times1{\cdot}25\times{\cdot}25}{3\,!}\times{\cdot}0054+\frac{2{\cdot}25\times1{\cdot}25\times{\cdot}25\times{\cdot}75}{4\,!}\times{\cdot}0022$$

$$=16{\cdot}9216.$$

Example 3. From the following table find Sun's Right Ascension for April 20, 1898.

Date	Sun's R. A.			Δ		Δ^2	Δ^3	Δ^4
1898	H.	M.	S.	M.	S.	S.	S.	S.
April 16	1	38	19·59					
				18	36·25			
21	1	56	55·84			10·99		
				18	47·24		1·05	
26	2	15	43·08			12·04		0·01
				18	59·28		1·06	
May 1	2	34	42·36			13·10		
				19	12·88			
6	2	53	54·74					

Here $a =$ April 16, $\omega = 5$ and $x = \cdot 8$.

		H. M. S.		H. M. S.
∴ Sun's R.A. for April 20 =		1 38 19·59 =		1 38 19·59
	+0·80 ×	18 36·25	+0	14 53·000
	−0·08 ×	10·99	−	0·879
	+0·032 ×	1·05	+	0·034
	−0·0176 ×	0·01	−	0·000
			=	1 53 11·75

∴ Sun's R. A. for April 20 = 1 53 11·75,

which is the value given in the American Ephemeris for 1898.

4.3. Missing Terms. Suppose the given values of the function do not form a complete series of equidistant values in consequence of the absence of one or more intermediate terms. Let it be required to supply the missing terms.

Let $f(a)$, $f(a+\omega)$, $f(a+2\omega)$, ... denote the complete series, of which n terms are supposed given. If one term is missing, the number of terms in the complete series is $n+1$, and to find the missing term put $\Delta^n f(a) = 0$. Then by §1.10 we have

$$(1)\quad f(a+n\omega) - nf(a+\overline{n-1}\omega) + \frac{n(n-1)}{2!} f(a+\overline{n-2}\omega) - \dots + (-1)^n f(a) = 0$$

from which any missing term is expressed in terms of the given n values of $f(x)$. Thus to interpolate a term midway between two others we have

$$f(a+2\omega) - 2\, f(a+\omega) + f(a) = 0,$$

so that

$$f(a+\omega) = \tfrac{1}{2}\{f(a) + f(a+2\omega)\}.$$

To supply a middle term in a series of five, we have

$$f(a) - 4f(a+\omega) + 6f(a+2\omega) - 4f(a+3\omega) + f(a+4\omega) = 0,$$

so that

$$f(a+2\omega) = \tfrac{1}{6}[4\{f(a+\omega) + f(a+3\omega\} - \{f(a) + f(a+4\omega)\}].$$

If two terms are missing and n terms are given we must also put

$$\Delta^n f(a+\omega) = 0,$$

$$(2)\qquad i.e.,\ f(a+\overline{n+1}\omega) - nf(a+n\omega) + \frac{n(n-1)}{2!} f(a+\overline{n-1}\omega) - \dots\dots + (-1^n) f(a+\omega) = 0.$$

From the equations (1) and (2) we evaluate the two unknown quantities.

In general, if r terms are missing and n terms are given, we require the following r equations to evaluate them :

$$\Delta^n f(a+\overline{r-1}\,\omega)=0, \qquad r=1,\,2,\,3,\ldots,r.$$

Example 1. Given $\log_{10}121=2{\cdot}0827854$, $\log_{10}124=2{\cdot}0934217$, $\log_{10}122=2{\cdot}0863598$, $\log_{10}125=2{\cdot}0969100$, find the value of $\log_{10}123$.

Putting $\Delta^4 f(121)=0$, we have

$$\begin{aligned}\log_{10}123&=\tfrac{1}{6}\{4(\log_{10}122+\log_{10}124)-(\log_{10}121+\log_{10}125)\}\\&=\tfrac{1}{6}\{16{\cdot}7191260-4{\cdot}1796954\}\\&=2{\cdot}0899051.\end{aligned}$$

Example 2. Complete the following table

$x=2{\cdot}0$	$2{\cdot}1$	$2{\cdot}2$	$2{\cdot}3$	$2{\cdot}4$	$2{\cdot}5$	$2{\cdot}6$
$f(x)=\ {\cdot}135$		${\cdot}111$	${\cdot}100$		${\cdot}082$	${\cdot}074$

Here five entries are known and hence the equations are

$$f(2{\cdot}5)-5f(2{\cdot}4)+10f(2{\cdot}3)-10f(2{\cdot}2)+5f(2{\cdot}1)-f(2{\cdot}0)=0,$$

$$f(2{\cdot}6)-5f(2{\cdot}5)+10f(2{\cdot}4)-10f(2{\cdot}3)+5f(2{\cdot}2)-f(2{\cdot}1)=0.$$

Substituting the values of $f(x)$ and solving we get

$$f(2{\cdot}1)={\cdot}123 \text{ and } f(2{\cdot}4)={\cdot}090.$$

4.4. Subdivision of Intervals. A frequent problem in actuarial work is the interpolation for values of a function $f(x)$ between every two consecutive terms of a given series. For example, premiums of policies maturing at every fifth year or tenth year being given, it may be required to find the premiums for every consecutive year. This operation is known as *subtabulation*.

Let the given series be $f(a)$, $f(a+\omega)$, $f(a+2\omega),\ldots$ and suppose that it is required to divide the interval ω into m subintervals so that $(m-1)$ intermediate values are to be interpolated between every two given consecutive entries.

We have by Gregory-Newton formula

$$(1)\qquad f\left(a+\frac{k\omega}{m}\right)=E^{\frac{k}{m}}f(a)=(1+\Delta)^{\frac{k}{m}}f(a)$$

$$=f(a)+\frac{k}{m}\Delta f(a)+\frac{1}{2!}\,\frac{k}{m}\left(\frac{k}{m}-1\right)\Delta^2 f(a)+\ldots\ldots$$

and putting $k=1, 2, 3, \ldots\ldots, m-1$ in succession we get the $m-1$ entries between $f(a)$ and $f(a+\omega)$. The values of $\Delta f(a)$, $\Delta^2 f(a)$, etc., are obtained from the given series of values. Next taking k from $m+1$ to $2m-1$, we get $m-1$ terms between $f(a+\omega)$ and $f(a+2\omega)$, and so on.

But this process is laborious for calculation of a series of values. It is better first to obtain the differences of the new sequence of values of the function, and then to calculate them from those differences.

Let Δ_1 be the difference operator for the new sequence of terms

$$(2) \qquad f(a),\ f\left(a+\frac{\omega}{m}\right),\ f\left(a+\frac{2\omega}{m}\right),\ \ldots\ldots,\ f(a+\omega), \ldots\ldots;$$

then since $\qquad (1+\Delta)f(a)=f(a+\omega)=(1+\Delta_1)^m f(a)$,

the two operators Δ and Δ_1 are connected by the relation

$$(1+\Delta_1)^m \equiv 1+\Delta$$

$$(3) \quad i.e. \quad \Delta_1=\frac{1}{m}\Delta+\frac{1}{2!}\,\frac{1}{m}\left(\frac{1}{m}-1\right)\Delta^2 + \frac{1}{3!}\,\frac{1}{m}\left(\frac{1}{m}-1\right)\left(\frac{1}{m}-2\right)\Delta^3+\ldots\ldots$$

and $$\Delta_1^r=\left\{\frac{1}{m}\Delta+\frac{1}{2!}\,\frac{1}{m}\left(\frac{1}{m}-1\right)\Delta^2 + \frac{1}{3!}\,\frac{1}{m}\left(\frac{1}{m}-1\right)\left(\frac{1}{m}-2\right)\Delta^3+\ldots\ldots\right\}^r.$$

For practical purposes we can ignore higher differences beyond Δ^4.

Putting $r=1, 2, 3, 4, \ldots\ldots$, we obtain the differences of the new sequence in terms of the differences of the old sequence. Thus

$$\Delta_1=\frac{1}{m}\Delta+\frac{1-m}{2m^2}\Delta^2+\frac{(1-m)(1-2m)}{6m^3}\Delta^3 + \frac{(1-m)(1-2m)(1-3m)}{24m^4}\Delta^4,$$

$$\Delta_1^2=\frac{1}{m^2}\Delta^2+\frac{1-m}{m^3}\Delta^3+\frac{(1-m)(7-11m)}{12m^4}\Delta^4,$$

$$\Delta_1^3=\frac{1}{m^3}\Delta^3+\frac{3(1-m)}{2m^4}\Delta^4.$$

$$\Delta_1^4=\frac{1}{m^4}\Delta^4.$$

When the differences are thus calculated, sequence (2) is easily obtained by simple addition.

Suppose for example that $m=5$ and differences up to Δ^4 are retained.

We easily find that

$$\begin{aligned}\Delta_1&={\cdot}2\Delta-{\cdot}08\Delta^2+{\cdot}048\Delta^3-{\cdot}0336\Delta^4,\\ \Delta_1^2&={\cdot}04\Delta^2-{\cdot}032\Delta^3+{\cdot}0256\Delta^4,\\ \Delta_1^3&={\cdot}008\Delta^3-{\cdot}0096\Delta^4,\\ \Delta_1^4&={\cdot}0016\Delta^4.\end{aligned}$$

Example. Annual premiums for policies maturing at a fixed age are given for the following ages:—

Age	20	25	30	35	40
Premium	30·5	34·5	40	47·75	59·25.

Calculate the premiums for all ages from 20 to 40.

Forming the difference table for quinquennial entries, we have

		Δ	Δ^2	Δ^3	Δ^4
20	30·5				
		4			
25	34·5		1·5		
		5·5		·75	
30	40		2·25		·75
		7·75		1·5	
35	47·75		3·75		
		11·5			
40	59·25				

Hence

$$\begin{aligned}\Delta_1 f(20)&={\cdot}2\times4-{\cdot}08\times1{\cdot}5+{\cdot}048\times{\cdot}75-{\cdot}0336\times{\cdot}75\\ &={\cdot}6908\\ \Delta_1^2 f(20)&={\cdot}04\times1{\cdot}5-{\cdot}032\times{\cdot}75+{\cdot}0256\times{\cdot}75\\ &={\cdot}0552\\ \Delta_1^3 f(20)&={\cdot}008\times{\cdot}75-{\cdot}0096\times{\cdot}75\\ &=-{\cdot}0012\\ \Delta_1^4 f(20)&={\cdot}0016\times{\cdot}75={\cdot}0012.\end{aligned}$$

We assume fourth differences to be constant.

Hence completing the table of differences we have

		Δ_1	Δ_1^2	Δ_1^3	Δ_1^4
20	30·5				
		·6908			
21	31·1908		·0552		
		·7460		$-$·0012	
22	31·9368		·0540		·0012
		·8000		0	
23	32·7368		·0540		·0012
		·8540		·0012	
24	33·5908		·0552		·0012
		·9092		·0024	
25	34·5000		·0576		
		·9668			
26	35·4668				

and so on.

4.5. Method of checking the numerical accuracy of the Differences.

The algebraic sum of any r *consecutive values of* Δ^n *is equal to the last minus the first of the* $r+1$ *consecutive values of* Δ^{n-1} *used in forming the* r *values of* Δ^n.

Let the tabulated values for Δ^{n-1} and Δ^n be as follows:

$$\Delta^{n-1} : h_1 \quad h_2 \quad h_3 \ldots\ldots h_{r-1} \quad h_r \quad h_{r+1}.$$

$$\Delta^n : \quad k_1 \quad k_2 \quad k_3 \ldots\ldots k_{r-1} \; k_r.$$

Then $$k_1 = h_2 - h_1, \qquad k_2 = h_3 - h_2, \ldots\ldots$$

$$k_{r-1} = h_r - h_{r-1}, \quad k_r = h_{r+1} - h_r.$$

Hence by addition we have

$$k_1 + k_2 + \ldots\ldots + k_{r-1} + k_r = h_{r+1} - h_1.$$

For illustration take the four values of $\Delta_1{}^3$ in table of §4.4 (Ex.) and the first and last values of $\Delta_1{}^2$; then

$$-{·}0012 + 0 + {·}0012 + {·}0024 = {·}0576 - {·}0552 = {·}0024.$$

Similarly $${·}0552 + {·}0540 + {·}0540 + {·}0552 + {·}0576$$

$$= {·}9668 - {·}6908 = {·}2760.$$

4.6. Inversion of a Series of Functions. It will be sometimes necessary to invert the series of tabulated values of a function, *i.e.*, to bring the last value into the first position, the last but one into the second position, and so on. The effect of such inversion on the tabulated columns of differences will be that each column of differences will be inverted, like the column of functions itself, and that the odd order differences will change signs throughout, while the even order differences will remain unaltered in sign.

4.7. Lagrange's Interpolation Formula. *Given n values of a function which are not consecutive and equidistant, to find the general expression for the function.*

Let the values of $f(x)$ be given for $x=x_1, x_2, \ldots, x_n$.

Assuming $f(x)$ to be a rational integral function of x of degree $n-1$, it can be put in the form

$$f(x)=A_1(x-x_2)(x-x_3)\ldots\ldots(x-x_n)+A_2(x-x_1)(x-x_3)\ldots\ldots(x-x_n)$$
$$+\ldots\ldots+A_n(x-x_1)(x-x_2)\ldots\ldots(x-x_{n-1}),$$

each of the n terms wanting one of the factors $x-x_1, x-x_2, \ldots\ldots, x-x_n$, and each is multiplied by an unknown constant. To evaluate the A's, put $x=x_1$, then

$$f(x_1)=A_1(x_1-x_2)(x_1-x_3)\ldots\ldots(x_1-x_n)$$

or

$$A_1=\frac{f(x_1)}{(x_1-x_2)(x_1-x_3)\ldots\ldots(x_1-x_n)}.$$

Similar expressions are obtained for $A_2, A_3, \ldots\ldots, A_n$. Thus we get

$$(1)\qquad f(x)=\frac{(x-x_2)(x-x_3)\ldots\ldots(x-x_n)}{(x_1-x_2)(x_1-x_3)\ldots\ldots(x_1-x_n)}f(x_1)$$
$$+\frac{(x-x_1)(x-x_3)\ldots\ldots(x-x_n)}{(x_2-x_1)(x_2-x_3)\ldots\ldots(x_2-x_n)}f(x_2)$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+\frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\ldots\ldots(x_n-x_{n-1})}f(x_n).$$

This is known as Lagrange's[*] Formula of Interpolation.

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[*] *Journal de l'Ecole Polytechnique*, II, 277.

Putting $$\phi(x)=(x-x_1)(x-x_2)\ldots\ldots(x-x_n),$$
it can be put as

$$f(x)=\Sigma\frac{f(x_r)}{(x-x_r)}\cdot\frac{\phi(x)}{\phi'(x_r)}.\quad r=1,\ 2,\ \ldots\ldots,\ n.$$

Example 1. From the following data find the expression for $f(x)$ and also the value of $f(10)$.

$$\begin{array}{lcccc} x & : 0, & 1, & 3, & 7 \\ f(x): & 5, & 7, & 47, & 439. \end{array}$$

Applying Lagrange's formula, we have

$$f(x)=\frac{(x-1)(x-3)(x-7)}{(0-1)(0-3)(0-7)}.5+\frac{x(x-3)(x-7)}{1(1-3)(1-7)}.7$$

$$+\frac{x(x-1)(x-7)}{3(3-1)(3-7)}.47+\frac{x(x-1)(x-3)}{7(7-1)(7-3)}.439$$

$$=x^3+2x^2-x+5.$$

Putting $x=10$, we have $f(10)=1195$.

The value of $f(10)$ is also obtained by putting $x=10$ in §4.7(1).

Example 2. Two observations of a certain quantity were made at the interval of a day from each other, the first gave for its value a and the second $-b$. When was its value zero?

Here we are given $f(0)=a$ and $f(1)=-b$, and it is required to find x for which $f(x)=0$.

We have $$f(x)=\frac{(x-1)}{0-1}a+\frac{(x-0)}{1-0}(-b).$$

Hence $$0=a(x-1)+bx \quad\text{or}\quad x=\frac{a}{a+b},$$

which is the time in fractions of a day from the first observation to the moment required.

The result is also obtained from §4.2(2)

$$f(x)=f(0)+x\Delta f(0).$$

As there are only two observations we take $\Delta^2 f(0)$, etc., to be zero.

Example 3. Three observations of a quantity are made near its time of maximum or minimum. Determine the time of its maximum or minimum.

Let a, b, c represent the times of observation and $f(x)$ the magnitude of the quantity at the time x, so that $f(a)$, $f(b)$ and $f(c)$ are the observed values. By Lagrange's formula we have

$$f(x)=\frac{(x-b)(x-c)}{(a-b)(a-c)}f(a)+\frac{(x-c)(x-a)}{(b-c)(b-a)}f(b)+\frac{(x-a)(x-b)}{(c-a)(c-b)}f(c).$$

As this function of x is to be a maximum or minimum, we equate to zero the differential coefficient of $f(x)$ with respect to x. Thus we get

$$x=\frac{(b^2-c^2)f(a)+(c^2-a^2)f(b)+(a^2-b^2)f(c)}{2\{(b-c)f(a)+(c-a)f(b)+(a-b)f(c)\}},$$

the value of x at which $f(x)$ is a maximum or minimum. By this formula the meridian altitude of the sun or of a star may be found when an observation precisely on the meridian cannot be made.

4.8. Irregularities in Differences. If, instead of taking the actual numerical values in the table, approximate values are taken to certain given places of decimals, the result will be that the differences will exhibit a small degree of *irregularity* owing to the omission of decimals in the approximate values of the functions employed. No such irregularity will appear if actual values are taken, so that the differences then will be perfectly smooth or *regular*. Thus taking a function of the nth degree we know that Δ^n will be constant, but when approximate values are taken, the values of Δ^n for different arguments will slightly vary. Thus a slight deviation from the true value of a tabular function will manifest itself by means of irregularities in differences.

4.9. Accidental Errors. If however one value of a series is in error by an appreciable quantity, we can locate the error and its magnitude from an inspection of the table.

Let the values of the function $f(x)$ be tabulated for values of $x=a$, $a+1$,...... and suppose that $f(a+6)$ is in error by the quantity ϵ, so that $f(a+6)+\epsilon$ is tabulated in place of the true value $f(a+6)$. The errors in the table will be manifest thus:

	Δ	Δ^2	Δ^3	Δ^4	Δ^5
$f(a)$					
	$\Delta f(a)$				
$f(a+1)$		$\Delta^2 f(a)$			
	$\Delta f(a+1)$		$\Delta^3 f(a)$		
$f(a+2)$		$\Delta^2 f(a+1)$		$\Delta^4 f(a)$	
	$\Delta f(a+2)$		$\Delta^3 f(a+1)$		$\Delta^5 f(a)$
$f(a+3)$		$\Delta^2 f(a+2)$		$\Delta^4 f(a+1)$	
	$\Delta f(a+3)$		$\Delta^3 f(a+2)$		$\Delta^5 f(a+1)+\epsilon$
$f(a+4)$		$\Delta^2 f(a+3)$		$\Delta^4 f(a+2)+\epsilon$	
	$\Delta f(a+4)$		$\Delta^3 f(a+3)+\epsilon$		$\Delta^5 f(a+2)-5\epsilon$
$f(a+5)$		$\Delta^2 f(a+4)+\epsilon$		$\Delta^4 f(a+3)-4\epsilon$	
	$\Delta f(a+5)+\epsilon$		$\Delta^3 f(a+4)-3\epsilon$		$\Delta^5 f(a+3)+10\epsilon$
$f(a+6)+\epsilon$		$\Delta^2 f(a+5)-2\epsilon$		$\Delta^4 f(a+4)+6\epsilon$	
	$\Delta f(a+6)-\epsilon$		$\Delta^3 f(a+5)+3\epsilon$		$\Delta^5 f(a+4)-10\epsilon$
$f(a+7)$		$\Delta^2 f(a+6)+\epsilon$		$\Delta^4 f(a+5)-4\epsilon$	
	$\Delta f(a+7)$		$\Delta^3 f(a+6)-\epsilon$		$\Delta^5 f(a+5)+5\epsilon$
$f(a+8)$		$\Delta^2 f(a+7)$		$\Delta^4 f(a+6)+\epsilon$	
	$\Delta f(a+8)$		$\Delta^3 f(a+7)$		$\Delta^5 f(a+6)-\epsilon$
$f(a+9)$		$\Delta^2 f(a+8)$		$\Delta^4 f(a+7)$	
	$\Delta f(a+9)$		$\Delta^3 f(a+8)$		$\Delta^5 f(a+7)$
$f(a+10)$		$\Delta^2 f(a+9)$		$\Delta^4 f(a+8)$	
	$\Delta f(a+10)$		$\Delta^3 f(a+9)$		
$f(a+11)$		$\Delta^2 f(a+10)$			
	$\Delta f(a+11)$				
$f(a+12)$					

From an inspection of the table we see that differences lying along and inside the lines sloping upwards and downwards from the incorrect tabulated value are also in error. To the regular values are alternately added and subtracted terms in ϵ. The law of progression and increase in the coefficients of ϵ along the successive orders of differences is that of binomial coefficients with alternate signs. We have only to carry on differencing to that order at which the differences of the correct functions would vanish or sensibly so. The location and magnitude of the error will then be clearly shewn by a succession of positive and negative terms with binomial coefficients.

Thus if the values of Δ^4 vanish in the correct table, the fourth differences in the incorrect table will be $0, +\epsilon, -4\epsilon, +6\epsilon, -4\epsilon, +\epsilon, 0$. The initial term $+\epsilon$ is the error sought. The incorrect function is then obtained by tracing backward along the line of downward slope. Thus in the table we come to $f(a+6)+\epsilon$. The correction is the negative of the error ϵ. Hence the true value is $f(a+6)$.

Example 1. Find the error in the following table of $f(x)=x^3$.

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4	C	Δ^4+C
1	1						
		7					
2	8		12				
		19		6			
3	27		18		0		0
		37		6			
4	64		24		−10	+10	0
		61		−4			
5	125		20		+40	−40	0
		81		+36			
6	206		56		−60	+60	0
		137		−24			
7	343		32		+40	−40	0
		169		16			
8	512		48		−10	+10	0
		217		6			
9	729		54				
		271					
10	1000						

Here the true value of Δ^4 is zero but in the table we have a complete alternation of signs and it is easily seen that the tabulated values for Δ^4 are the binomial coefficients of the fourth order multiplied by 10. Further as the first incorrect entry is -10, the correction is $+10$, and tracing back downwards from -10 we come to the required incorrect entry 206. Hence the true value is 216. The same is also obtained by tracing back upwards from the last incorrect entry -10 in the column for Δ^4.

In the column C are given corrections to Δ^4 due to correction $+10$ to $f(6)$ and the last column gives true values of Δ^4.

4.10. When two or more consecutive and neighbouring entries are in error, the detection becomes difficult and complicated, and in some cases even impossible. The entries may all contain systematic errors which do not affect the regularity of the differences. It is possible to detect such errors only when two or three entries are in error of accidental character which produce a distinct and definite irregularity in the difference table.

Example. Correct the errors in the following table of $f(x)=x^3-5x+12$.

x	$f(x)$	Δ	Δ^2	Δ^3	C_1	Δ^3+C_1	C_2	$\Delta^3+C_1+C_2$
1	8							
		2						
2	10		12					
		14		6		6		6
3	24		18					
		32		14	-8	6		6
4	56		32					
		64		-18	24	6		6
5	120		14					
		78		20	-24	-4	10	6
6	198		34					
		112		28	8	36	-30	6
7	310		62					
		174		-24		-24	30	6
8	484		38					
		212		16		16	-10	6
9	696		54					
		266						
10	962							

Here the third difference is constant $(=6)$ and the first incorrect entry is 14. Hence in the column C_1 we put the correction -8, and following the law of binomial coefficients of the third order with alternate signs, the corrections are -8, 24, -24, 8. The column Δ^3+C_1 gives the corrected series for the error 8, but here also the first incorrect entry is -4, which suggests a further correction of 10. The column C_2 gives corrections for the error -10. The final corrected column $\Delta^3+C_1+C_2$ gives the true value 6 of the third differences. The incorrect entries are traced to be $f(5)=120$ and $f(7)=310$, which should be $f(5)=120-8=112$ and $f(7)=310+10=320$.

4.11. If the error begins in the column of earlier differences, higher differences of some order will indicate corrections corresponding to the binomial coefficients of a lower order than that of the differences in question. Thus if Δ^5 requires a correction of the order 1, 3, 3, 1, it is clear that an error exists in Δ^2, of which Δ^5 is the third difference. In general, when Δ^n requires a correction of the rth order, an error will exist in Δ^{n-r}.

When either the first or the last function of a series is in error, only the first or the last term in each order of differences will be in error by the same amount. In such cases, methods explained above are of little assistance in correcting the errors.

4.12. Mechanical Quadrature. One of the chief applications of interpolation formulæ is the evaluation of definite integrals, *i.e.*, the determination of the areas of curves bounded by the two extreme ordinates, the x-axis and the curve. When the equation of the curve is known and integrable, the result is obtained by direct integration. But in physical and natural phenomena the above conditions are hardly satisfied. Thus astronomical observations, rates of mortality, etc., are obtained from actual observations and the expression for the function corresponding to these data is not known. The process by which such areas are calculated without a knowledge of the form of function is known as *Numerical Integration* or *Mechanical Quadrature*.

Suppose $n+1$ equidistant ordinates $f(0)$, $f(1)$, $f(2)$, ... $f(n)$ are given and it is required to find an approximate value of $\int_0^n f(x)dx$.

We have, by Gregory-Newton formula of interpolation,

$$f(x)=f(0)+x\Delta f(0)+\frac{x^{(2)}}{2!}\Delta^2 f(0)+\frac{x^{(3)}}{3!}\Delta^3 f(0)+\dots\dots+\frac{x^{(n)}}{n!}\Delta^n f(0).$$

Hence

$$\int_0^n f(x)dx=f(0)\int_0^n dx+\Delta f(0)\int_0^n x\,dx+\frac{\Delta^2 f(0)}{2!}\int_0^n x^{(2)}dx$$

$$+\frac{\Delta^3 f(0)}{3!}\int_0^n x^{(3)}dx+\dots\dots$$

$$(1)\quad =nf(0)+\frac{n^2}{2}\Delta f(0)+\left(\frac{n^3}{3}-\frac{n^2}{2}\right)\frac{\Delta^2 f(0)}{2!}+\left(\frac{n^4}{4}-n^3+n^2\right)\frac{\Delta^3 f(0)}{3!}$$

$$+\left(\frac{n^5}{5}-\frac{3n^4}{2}+\frac{11n^3}{3}-3n^2\right)\frac{\Delta^4 f(0)}{4!}+\dots\dots$$

As $n+1$ values of $f(x)$ are known, differences up to Δ^n are calculated from the given data and higher differences can be neglected. The expression (1) gives an approximate value of the integral.

4.13. Particular Cases.

(1) Simpson's Rule. Suppose that three entries $f(0)$, $f(1)$ and $f(2)$ are only given. Then retaining terms as far as the second difference we have

$$\int_0^2 f(x)dx = 2f(0) + 2\Delta f(0) + \frac{1}{3}\Delta^2 f(0).$$

Substituting the values of Δ and Δ^2 we have

$$\int_0^2 f(x)dx = \frac{1}{3}\{f(0) + 4f(1) + f(2)\}.$$

If the third differences of $f(x)$ are constant, Simpson's rule gives the exact value.

(2) The Three-eighths Rule. If four equidistant ordinates are given we have

$$\int_0^3 f(x)dx = \frac{3}{8}\{f(0) + 3f(1) + 3f(2) + f(3)\}.$$

(3) Weddle's Rule.[*] Suppose that seven equidistant ordinates are given. Then

$$\int_0^6 f(x)dx = 6f(0) + 18\Delta f(0) + 27\Delta^2 f(0) + 24\Delta^3 f(0) + \frac{123}{10}\Delta^4 f(0)$$
$$+ \frac{33}{10}\Delta^5 f(0) + \frac{41}{140}\Delta^6 f(0).$$

Transforming the Δ's in terms of the ordinates it will be found that the coefficients of the terms are large. This is due to the awkward coefficient $\frac{41}{140}$ of $\Delta^6 f(0)$. As the sixth differences are small, the error involved in replacing $\frac{41}{140}$ by $\frac{42}{140}$, i.e., by $\frac{3}{10}$ will be negligible, but the formula will be much simplified. The new formula will involve an error of $\frac{1}{140}\Delta^6 f(0)$ and will be strictly correct up to the fifth differences but virtually correct to the seventh differences. On reduction it becomes

$$\int_0^6 f(x)dx = \frac{3}{10}\Big[\{f(0) + f(6)\} + 5\{f(1) + f(5)\} + \{f(2) + f(4)\} + 6f(3)\Big].$$

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[*] *Math. Journal*, ix, p. 79.

4.14. Change of Unit and of Origin. The formulæ of §4.12 and §4.13 have been obtained by taking the interval to be unity. If however the interval be ω, the given values of $f(x)$ will be $f(0)$, $f(\omega)$, $f(2\omega)$,....... Hence taking the new variable to be x' we have

$$x'=\omega x \quad \text{and} \quad \therefore \quad dx'=\omega dx.$$

The new limits of integration will be 0 and $n\omega$.

Thus Simpson's formula will transform into

$$\frac{1}{\omega}\int_0^{2\omega} f(x')\,dx'=\frac{1}{3}\left\{f(0)+4f(\omega)+f(2\omega)\right\}.$$

or changing the variable to x we have

$$(1) \qquad \frac{1}{\omega}\int_0^{2\omega} f(x)dx=\frac{1}{3}\left\{f(0)+4f(\omega)+f(2\omega)\right\}.$$

Similarly Weddle's rule for the interval ω will be

$$\frac{1}{\omega}\int_0^{6\omega} f(x)dx=\frac{3}{10}\left\{f(0)+5f(\omega)+f(2\omega)+6f(3\omega)+f(4\omega)+5f(5\omega)+f(6\omega)\right\}.$$

Any formula of approximate integration for unit interval can thus be transformed to one for interval ω.

A change of origin from $x=0$ to $x=a$ is obtained by substituting $f(a)$, $f(a+\omega)$, $f(a+2\omega)$....... in place of $f(0)$, $f(\omega)$, $f(2\omega)$, etc., in the formula. Thus in Simpson's formula, if the origin is transferred to $x=a$, we will get

$$\frac{1}{\omega}\int_a^{a+2\omega} f(x)dx=\frac{1}{3}\left\{f(a)+4f(a+\omega)+f(a+2\omega)\right\}.$$

This formula was obtained by James Gregory* in 1668 and by Thomas Simpson† in 1743.

If the origin is transferred to $x=\omega$ and the same area as that given by (1) is required, the transformed formula will be

$$\frac{1}{\omega}\int_{-\omega}^{\omega} f(x)dx=\frac{1}{3}\left\{f(-\omega)+4f(0)+f(\omega),\right\}.$$

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* **James Gregory**—*Exercit. Geom.*, London.

† **Thomas Simpson**—*Math. Dissertations*, London (1743), p. 109.

Similarly the Three-eighths rule becomes

$$\frac{1}{\omega}\int_a^{a+3\omega} f(x)dx=\frac{3}{8}\left\{f(a)+3f(a+\omega)+3f(a+2\omega)+f(a+3\omega)\right\}.$$

4.15. Extension of Simpson's Rule.

We have $\int_0^2 f(x)dx=\frac{1}{3}\left\{f(0)+4f(1)+f(2)\right\}.$

Changing the origin

$$\int_2^4 f(x)dx=\frac{1}{3}\left\{f(2)+4f(3)+f(4)\right\}.$$

Similarly $\int_4^6 f(x)dx=\frac{1}{3}\left\{f(4)+4f(5)+f(6)\right\}.$

……………………………………………………

$$\int_{2n-2}^{2n} f(x)dx=\frac{1}{3}\left\{f(2n-2)+4f(2n-1)+f(2n)\right\}.$$

Hence by addition

$$(1)\quad \int_0^{2n} f(x)dx=\frac{1}{3}\left\{f(0)+4f(1)+2f(2)+4f(3)+2f(4)+\ldots\ldots+f(2n)\right\}.$$

This formula gives better result than the formula for the interval n,

$$\int_0^{2n} f(x)dx=\frac{n}{3}\left\{f(0)+4f(n)+f(2n)\right\},$$

and is extensively used in practical mathematics.

If the origin is transferred to a and the interval be ω, we get (1) in the form

$$(2)\quad \frac{1}{\omega}\int_a^{a+2n\omega} f(x)\,dx=\frac{1}{3}\Big[f(a)+2\left\{f(a+2\omega)+f(a+4\omega)+\ldots\ldots\right\}$$
$$+4\left\{f(a+\omega)+f(a+3\omega)+\ldots\ldots\right\}+f(a+2n\omega)\Big].$$

Example. Find the value of $\int_{20°}^{44°}\cos\theta\, d\theta$ from the following data:

θ	$\cos\theta$	θ	$\cos\theta$
20°	0·939693	36°	0·809017
24°	0·913545	40°	0·766044
28°	0·882948	44°	0·719340
32°	0·848048		

(1) Taking only three values for 20°, 32° and 44° and applying Simpson's rule, we have

$$\int_{20^\circ}^{44^\circ} \cos\theta\, d\theta = \frac{\omega}{3}\left\{\cos 20^\circ + 4\cos 32^\circ + \cos 44^\circ\right\}$$

$$= \frac{12^\circ}{3}\left\{0\cdot939693 + 3\cdot392192 + 0\cdot719340\right\} \because \omega = 12^\circ$$

$$= \frac{\pi}{45} \times 5\cdot051225 = \cdot069813 \times 5\cdot051225$$

$$= \cdot35264117\ldots\ldots$$

(2) Generalised Simpson's rule gives

$$\int_{20^\circ}^{44^\circ} \cos\theta\, d\theta = \frac{4^\circ}{3}\left\{\cos 20^\circ + 4\cos 24^\circ + 2\cos 28^\circ + 4\cos 32^\circ + 2\cos 36^\circ + 4\cos 40^\circ + \cos 44^\circ\right\}$$

$$= \frac{4^\circ}{3} \times 15\cdot153511$$

$$= \cdot352638189\ldots\ldots$$

(3) The Three-eighths rule with interval of differencing 8° gives

$$\int_{20^\circ}^{44^\circ} \cos\theta\, d\theta = \frac{8^\circ \times 3}{8}\left\{\cos 20^\circ + 3\cos 28^\circ + 3\cos 36^\circ + \cos 44^\circ\right\}$$

$$= 3^\circ \times 6\cdot734928 = \cdot352640\ldots\ldots$$

(4) Weddle's rule gives

$$\int_{20^\circ}^{44^\circ} \cos\theta\, d\theta = \frac{3}{10}\cdot 4^\circ \left\{\cos 20^\circ + 5\cos 24^\circ + \cos 28^\circ + 6\cos 32^\circ + \cos 36^\circ + 5\cos 40^\circ + \cos 44^\circ\right\}$$

$$= 4^\circ \times \cdot3 \times 16\cdot837231$$

$$= \cdot35263813.$$

The true value of the definite integral is

$$\int_{20^\circ}^{44^\circ} \cos\theta\, d\theta = \sin 44^\circ - \sin 20^\circ$$

$$= 0\cdot694658 - 0\cdot342020 = 0\cdot352638.$$

Thus the values obtained by Generalised Simpson's rule and Weddle's rule are correct up to 6 places of decimals. Those given by Simpson's rule and Three-eighths rule are in slight error in the 5th or 6th place of decimals, being slightly in excess of the true value.

4.16. G. F. Hardy's Formulæ. Suppose the fifth differences of $f(x)$ are constant, and values of $f(x)$ for $x=0, 1, 3, 5, 6$ are given. $f(x)$ can evidently be represented by a polynomial of degree 5. Put

$$f(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5.$$

Changing the origin to -3,

$$\int_0^6 f(x)dx \quad \text{becomes} \quad \int_{-3}^3 f(x)dx$$

with the values of $f(x)$ for $x=-3, -2, 0, 2, 3$.

Now $$f(0)=a_0,\ f(-2)+f(2)=2a_0+8a_2+32a_4,$$

$$f(-3)+f(3)=2a_0+18a_2+162a_4,$$

and

(1) $$\int_{-3}^3 f(x)dx=6a_0+18a_2+97{\cdot}2a_4.$$

Now let us see if (1) can be expressed in the form

(2) $$\lambda f(0)+\mu\{f(-2)+f(2)\}+\nu\{f(-3)+f(3)\}.$$

Equating the coefficients in (1) and (2), we have

$$\lambda=2{\cdot}2,\ \mu=1{\cdot}62 \text{ and } \nu={\cdot}28,$$

so that

$$\int_{-3}^3 f(x)dx=2{\cdot}2f(0)+1{\cdot}62\{f(-2)+f(2)\}+{\cdot}28\{f(-3)+f(3)\},$$

(3) or $$\int_0^6 f(x)dx={\cdot}28\{f(0)+f(6)\}+1{\cdot}62\{f(1)+f(5)\}+2{\cdot}2f(3),$$

which is Hardy's Formula* (37).

If the interval of differencing be ω, this becomes

$$\int_0^{6\omega} f(x)dx=\omega[{\cdot}28\{f(0)+f(6\omega)\}+1{\cdot}62\{f(\omega)+f(5\omega)\}+2{\cdot}2f(3\omega)].$$

Similarly

$$\int_{6\omega}^{12\omega} f(x)dx=\omega[{\cdot}28\{f(6\omega)+f(12\omega)\}+1{\cdot}62\{f(7\omega)+f(11\omega)\}+2{\cdot}2f(9\omega)],$$

and so on.

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* See G. King—*Life Contingencies.*

Hence

$$(4)\qquad \int_0^x f(x)dx=\omega[\cdot28\{f(0)+2f(6\omega)+2f(12\omega)+\dots\}$$
$$+1\cdot62\{f(\omega)+f(5\omega)+f(7\omega)+\dots\}$$
$$+2\cdot2\{f(3\omega)+f(9\omega)+f(15\omega)+\dots\}].$$

This is Hardy's Formula (38).

In problems of mortality table, ω is so chosen that 7ω is near about 100 ; (4) is then put in the modified form

$$(5)\qquad \int_0^x f(x)dx=\omega\{\cdot28f(0)+1\cdot62f(\omega)+2\cdot2f(3\omega)+1\cdot62f(5\omega)$$
$$+\cdot56f(6\omega)+1\cdot62f(7\omega)\}.$$

This is Hardy's Formula (39*a*).

Example. Evaluate $\displaystyle\int_0^{\frac{1}{2}}\frac{dx}{\sqrt{1-x^2}}$.

Here $f(x)=\dfrac{1}{\sqrt{1-x^2}}$. Take $\omega=\dfrac{1}{12}$.

$$\text{Then}\quad \int_0^{\frac{1}{2}}\frac{dx}{\sqrt{1-x^2}}=\frac{1}{12}\{\cdot28\times2\cdot1547005+1\cdot62\times2\cdot1035285$$
$$+2\cdot2\times1\cdot0327955\}=0\cdot5235985.$$

The true value of the integral is $\frac{\pi}{6}$ or $0\cdot5235987$, so there is a slight error in the seventh place of decimals.

4.17. Laplace's Formula for Mechanical Quadrature.*

Since $E=e^D=1+\Delta$ $\qquad\therefore\ D=\log(1+\Delta)$.

Hence
$$\Delta=D\frac{\Delta}{\log(1+\Delta)}.$$

$$\therefore\quad \Delta\phi(x)=D\left\{1+\frac{\Delta}{2}-\frac{\Delta^2}{12}+\frac{\Delta^3}{24}-\dots\dots\right\}\phi(x).$$

or
$$\int\Delta\phi(x)dx=\left\{1+\frac{\Delta}{2}-\frac{\Delta^2}{12}+\frac{\Delta^3}{24}-\dots\dots\right\}\phi(x).$$

Since
$$\Big[\phi(x)\Big]_r^{r+1}=\Delta\phi(r)$$

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* *Mécanique Céleste*, IV, 207.

we get, on taking the limits 1 and 0 and putting $f(x)=\Delta\phi(x)$,

$$\int_0^1 f(x)dx=\left\{1+\frac{\Delta}{2}-\frac{\Delta^2}{12}+\frac{\Delta^3}{24}-\ldots\ldots\right\}f(0)$$

$$=\frac{f(0)+f(1)}{2}-\frac{\Delta^2 f(0)}{12}+\frac{\Delta^3 f(0)}{24}-\ldots\ldots$$

Similarly

$$\int_1^2 f(x)dx=\frac{f(1)+f(2)}{2}-\frac{\Delta^2 f(1)}{12}+\frac{\Delta^3 f(1)}{24}-\ldots\ldots$$

and so on.

Hence on addition of n such expressions we get

$$(1) \qquad \int_0^n f(x)dx=\frac{f(0)}{2}+f(1)+f(2)+\ldots\ldots+f(n-1)+\frac{f(n)}{2}$$

$$-\frac{1}{12}\left\{\Delta f(n)-\Delta f(0)\right\}$$

$$+\frac{1}{24}\left\{\Delta^2 f(n)-\Delta^2 f(0)\right\}$$

$$-\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

for

$$\Delta^r f(0)+\Delta^r f(1)+\ldots\ldots+\Delta^r f(n-1)$$

$$=\Delta^{r-1}\{\Delta f(0)+\Delta f(1)+\ldots\ldots+\Delta f(n-1)\}$$

$$=\Delta^{r-1}\{f(n)-f(0)\}.$$

Formula (1) labours under the disadvantage of containing differences of $f(n)$, which cannot be calculated from the values of $f(0)$, $f(1)$,......., $f(n)$.

A slightly modified result can be obtained thus:

$$\frac{\Delta}{\log(1+\Delta)}=\frac{-\Delta}{-\log E}=\frac{-E\Delta E^{-1}}{\log(1-\Delta E^{-1})}.$$

$$\therefore \quad \left\{1+\frac{\Delta}{2}-\frac{\Delta^2}{12}+\frac{\Delta^3}{24}-\ldots\ldots\right\}\phi(x)$$

$$=E\left\{1-\frac{\Delta E^{-1}}{2}-\frac{\Delta^2 E^{-2}}{12}-\frac{\Delta^3 E^{-3}}{24}-\ldots\ldots\right\}\phi(x).$$

Writing $f(x)$ for $\Delta\phi(x)$ and removing equal terms we have

$$-\frac{\Delta f(x)}{12}+\frac{\Delta^2 f(x)}{24}-\ldots\ldots=-\frac{\Delta f(x-1)}{12}-\frac{\Delta^2 f(x-2)}{24}-\ldots\ldots$$

Substituting the above in (1) we get

$$(2)\quad \int_0^n f(x)dx=\frac{f(0)}{2}+f(1)+f(2)+\ldots\ldots+f(n-1)+\frac{f(n)}{2}$$

$$-\frac{1}{12}\left\{\Delta f(n-1)-\Delta f(0)\right\}-\frac{1}{24}\left\{\Delta^2 f(n-2)+\Delta^2 f(0)\right\}$$

$$-\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Here all the differences can be calculated from the data given. This is a simple form of Gregory's formula §4.19(2).

4.18. Euler-Maclaurin Formula.* We have from §3.2(4)

$$\Sigma f(x)=C+\int f(x)dx-\frac{1}{2}f(x)+\frac{1}{12}\frac{df(x)}{dx}-\frac{1}{720}\frac{d^3f(x)}{dx^3}+\ldots\ldots$$

Taking this between the limits 0 and n, we get

$$\Sigma f(x)=f(0)+f(1)+f(2)+\ldots\ldots+f(n-1)$$

$$=\int_0^n f(x)dx-\frac{1}{2}\left\{f(n)-f(0)\right\}+\frac{1}{12}\left\{f'(n)-f'(0)\right\}$$

$$-\frac{1}{720}\left\{f'''(n)-f'''(0)\right\}+\ldots\ldots$$

Hence

$$(1)\quad \int_0^n f(x)dx=\frac{1}{2}f(0)+f(1)+f(2)+\ldots\ldots+f(n-1)+\frac{1}{2}f(n)$$

$$-\frac{1}{12}\left\{f'(n)-f'(0)\right\}+\frac{1}{720}\left\{f'''(n)-f'''(0)\right\}-\ldots\ldots$$

The above is a simple form of Euler-Maclaurin expansion for unit interval.

Changing the origin to a and taking ω as the interval of x we obtain a more general formula:

$$(2)\quad \frac{1}{\omega}\int_a^{a+n\omega} f(x)dx=\frac{1}{2}f(a)+f(a+\omega)+f(a+2\omega)+\ldots\ldots$$

$$+f(a+\overline{n-1}\omega)+\frac{1}{2}f(a+n\omega)-\frac{\omega}{12}\left\{f'(a+n\omega)-f'(a)\right\}$$

$$+\frac{\omega^3}{720}\left\{f'''(a+n\omega)-f'''(a)\right\}-\frac{\omega^5}{30240}\left\{f^{v}(a+n\omega)-f^{v}(a)\right\}+\ldots\ldots$$

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* **Euler**—*Comm. Acad. Sci. Imp. Petrop.* vi, (1738), p. 68
Maclaurin—*Treatise of Fluxions* (1742), p. 672.

Here the terms involving derivatives and the integral are affected with ω (see §4.14). This is the basic formula for expressing a definite integral in terms of given ordinates. If the form of the function is known and differentiable and the integral can be evaluated by some method, the above formulæ can be used in finding out the sum of a given series. When the ordinates are given but the form of the function is not known, the derivatives can be expressed in terms of differences (see §4.19) and the integral can then be obtained.

If $f(x)$ be a polynomial, the Euler-Maclaurin series terminates and is exact.

Example 1. Evaluate $\int_1^2 \frac{dx}{x}$ correct to five places of decimals.

In Euler-Maclaurin formula §4.18(2), put $a=1$, $n=5$, $\omega=0{\cdot}2$ and $f(x)=\frac{1}{x}$. Then

$$\frac{1}{0{\cdot}2}\int_1^2 \frac{dx}{x}=\frac{1}{2}\cdot\frac{1}{1}+\frac{1}{1{\cdot}2}+\frac{1}{1{\cdot}4}+\frac{1}{1{\cdot}6}+\frac{1}{1{\cdot}8}+\frac{1}{2}\cdot\frac{1}{2}$$

$$-\frac{0{\cdot}2}{12}\left\{-\frac{1}{2^2}+\frac{1}{1^2}\right\}+\frac{0{\cdot}008}{720}\left\{-\frac{2.3}{2^4}+\frac{2.3}{1^4}\right\}-\dots$$

$$={\cdot}50000+{\cdot}83333+{\cdot}71429+{\cdot}62500+{\cdot}55556+{\cdot}25000$$

$$-{\cdot}01250+{\cdot}0000625-\dots$$

$$=3{\cdot}46574.$$

$$\therefore\quad \int_1^2 \frac{dx}{x}={\cdot}693148={\cdot}69315,$$

agreeing with the true value of $\log_e 2$ correct to five places of decimals.

Example 2. Calculate correctly to ten places of decimals the sum of

$$\frac{1}{100^2}+\frac{1}{102^2}+\frac{1}{104^2}+\dots+\frac{1}{200^2}.$$

Find also the sum of the series to infinity.

Here $a=100$, $\omega=2$, $n=50$ and $f(x)=\frac{1}{x^2}$. Hence from Euler-Maclaurin expansion we have

$$\frac{1}{2}\int_{100}^{200}\frac{dx}{x^2}=\frac{1}{2}\cdot\frac{1}{100^2}+\frac{1}{102^2}+\dots+\frac{1}{2}\cdot\frac{1}{200^2}$$

$$-\frac{1}{6}\left\{-\frac{2}{200^3}+\frac{2}{100^3}\right\}+\frac{1}{90}\left\{-\frac{24}{200^5}+\frac{24}{100^5}\right\}-\dots$$

$$\therefore \frac{1}{100^2}+\frac{1}{102^2}+\dots+\frac{1}{200^2}$$

$$=\frac{1}{2}\left\{\frac{1}{100}-\frac{1}{200}\right\}+\frac{1}{2}\left\{\frac{1}{100^2}+\frac{1}{200^2}\right\}$$

$$+\frac{1}{3}\left\{\frac{1}{100^3}-\frac{1}{200^3}\right\}-\frac{4}{15}\left\{\frac{1}{100^5}-\frac{1}{200^5}\right\}+\dots$$

$$=\cdot 0025+\cdot 0000625+\cdot 00000029166666-\cdot 00000000002583$$

$$=\cdot 0025627916408 3$$

which is the sum required.

Again to find the sum to infinity, we have

$$\frac{1}{100^2}+\frac{1}{102^2}+\frac{1}{104^2}+\dots\dots \textit{ ad infinitum}$$

$$=\frac{1}{2}\cdot\frac{1}{100}+\frac{1}{2}\cdot\frac{1}{100^2}+\frac{1}{3}\cdot\frac{1}{100^3}-\frac{4}{15}\cdot\frac{1}{100^5}+\dots$$

$$=\cdot 005+\cdot 00005+\cdot 0000003333333-\cdot 000000000266$$

$$=\cdot 0050503333067$$

which is the sum required.

4.19. Gregory's Formula of Numerical Integration. Here the terms involving derivatives in Euler-Maclaurin expansion are replaced by differences.

Since $e^{\omega D}=1+\Delta, \quad \therefore \ \omega D=\log(1+\Delta)$,

(1) hence $\omega^r D^r=\left\{\Delta-\frac{\Delta^2}{2}+\frac{\Delta^3}{3}-\frac{\Delta^4}{4}+\dots\right\}^r$.

Thus $\omega f'(a)=\Delta f(a)-\frac{1}{2}\Delta^2 f(a)+\frac{1}{3}\Delta^3 f(a)-\frac{1}{4}\Delta^4 f(a)+\frac{1}{5}\Delta^5 f(a)-\dots$

$$\omega^2 f''(a)=\Delta^2 f(a)-\Delta^3 f(a)+\frac{11}{12}\Delta^4 f(a)-\frac{5}{6}\Delta^5 f(a)+\dots$$

$$\omega^3 f'''(a)=\Delta^3 f(a)-\frac{3}{2}\Delta^4 f(a)+\frac{7}{4}\Delta^6 f(a)-\dots$$

$$\omega^4 f''''(a)=\Delta^4 f(a)-2\Delta^5 f(a)+\dots$$

$$\omega^5 f'''''(a)=\Delta^5 f(a)-\frac{5}{2}\Delta^6 f(a)+\dots$$

Again reversing the order of entires, *i.e.*. forming a table in the order $f(a+n\omega)$, $f(a+\overline{n-1}\omega),\dots f(a)$ and applying (1), we have

$$\omega f'(a+n\omega)=\Delta f(a+\overline{n-1}\omega)+\frac{1}{2}\Delta^2 f(a+\overline{n-2}\omega)+\frac{1}{3}\Delta^3 f(a+\overline{n-3}\omega)$$

$$+\frac{1}{4}\Delta^4 f(a+\overline{n-4}\omega)+\frac{1}{5}\Delta^5 f(a+\overline{n-5}\omega)+\dots$$

$$\omega^2 f''(a+n\omega)=\Delta^2 f(a+\overline{n-2}\omega)+\Delta^3 f(a+\overline{n-3}\omega)+\tfrac{11}{12}\Delta^4 f(a+\overline{n-4}\omega)+\ldots$$

$$\omega^3 f'''(a+n\omega)=\Delta^3 f(a+\overline{n-3}\omega)+\tfrac{3}{2}\Delta^4 f(a+\overline{n-4}\omega)$$
$$+\tfrac{7}{4}\Delta^5 f(a+\overline{n-5}\omega)+\ldots$$

and so on.

Substituting these values of the derivates in Euler-Maclaurin expansion we obtain

$$\frac{1}{\omega}\int_a^{a+n\omega} f(x)dx=\tfrac{1}{2}f(a)+f(a+\omega)+\ldots+f(a+\overline{n-1}\omega)+\tfrac{1}{2}f(a+n\omega)$$

$$(2)\qquad -\frac{1}{12}\Big\{\Delta f(a+\overline{n-1}\omega)-\Delta f(a)\Big\}$$
$$-\frac{1}{24}\Big\{\Delta^2 f(a+\overline{n-2}\omega)+\Delta^2 f(a)\Big\}$$
$$-\frac{19}{720}\Big\{\Delta^3 f(a+\overline{n-3}\omega)-\Delta^3 f(a)\Big\}$$
$$-\frac{3}{160}\Big\{\Delta^4 f(a+\overline{n-4}\omega)+\Delta^4 f(a)\Big\}$$
$$-\frac{863}{60480}\Big\{(\Delta^5 f(a+\overline{n-5}\omega)-\Delta^5 f(a)\Big\}-\ldots$$

This is Gregory's Formula, discovered by him in 1670. It does not require data beyond $f(a+n\omega)$ to form the differences.

Example. Obtain the value of $\int_{20^\circ}^{44^\circ}\cos\theta\,d\theta$ from the data given in Ex. on p.105. The difference table is formed thus:

θ	$\cos\theta$	Δ	Δ^2	Δ^3	Δ^4
20°	0·939693				
		−·026148			
24°	0·913545		−·004449		
		−·030597		146	
28°	0·882948		− 4303		26
		− 34900		172	
32°	0·848048		− 4131		17
		− 39031		189	
36°	0·809017		− 3942		22
		− 42973		211	
40°	0·766044		− 3731		
		− 46704			
44°	0·719340				

Here $a=20^\circ$, $\omega=4^\circ=\frac{\pi}{45}$, $n=6$.

Hence substituting these values in §4·19(2), we have

$$\begin{array}{lrr} \frac{1}{4^\circ}\int_{20^\circ}^{44^\circ}\cos\theta\,d\theta=\cos 20^\circ+\cos 24^\circ+\dots+\cos 44^\circ= & & +5{\cdot}878635 \\ \qquad -\frac{1}{2}(\cos 20^\circ+\cos 44^\circ) & & -0{\cdot}829516.5 \\ \qquad -\frac{1}{12}(\Delta\cos 40^\circ-\Delta\cos 20^\circ) & + & 1713.0 \\ \qquad -\frac{1}{24}(\Delta^2\cos 36^\circ+\Delta^2\cos 20^\circ) & + & 340.8 \\ \qquad -\frac{19}{720}(\Delta^3\cos 32^\circ-\Delta^3\cos 20^\circ) & - & 1.7 \\ \qquad -\frac{3}{160}(\Delta^4\cos 28^\circ+\Delta^4\cos 20^\circ) & - & 0.9 \\ \hline & = + & 5{\cdot}051170 \end{array}$$

$$\therefore \int_{20^\circ}^{44^\circ}\cos\theta\,d\theta=\frac{\pi}{45}\times 5{\cdot}051170={\cdot}06981317\times 5{\cdot}051170$$

$$={\cdot}352638$$

which agrees with the true value given on p. 106.

4.20. Lubbock's Formula of Summation. Suppose that each interval in Euler-Maclaurin formula is subdivided into m parts, then the new formula is

$$(1)\ \frac{m}{\omega}\int_a^{a+n\omega}f(x)dx=\left\{f(a)+f(a+\frac{\omega}{m})+f(a+\frac{2\omega}{m})+\dots+f(a+n\omega)\right\}$$

$$-\tfrac{1}{2}\left\{f(a)+f(a+n\omega)\right\}-\frac{\omega}{12m}\left\{f'(a+n\omega)-f'(a)\right\}$$

$$+\frac{\omega^3}{720m^3}\left\{f'''a+n\omega)-f'''(a)\right\}-\dots$$

Eliminating the integral between (1) and the original formula §4.18(2) we get

$$(2) \qquad f(a)+f\left(a+\frac{\omega}{m}\right)+f\left(a+\frac{2\omega}{m}\right)+\ldots+f(a+n\omega)$$

$$=m\left\{f(a)+f(a+\omega)+\ldots+f(a+n\omega)\right\}-\frac{m-1}{2}\left\{f(a+n\omega)+f(a)\right\}$$

$$-\frac{m^2-1}{12m}\,\omega\left\{f'(a+n\omega)-f'(a)\right\}$$

$$+\frac{m^4-1}{m^3}\,\frac{\omega^3}{720}\left\{f'''(a+n\omega)-f'''(a)\right\}\ldots\ldots$$

This formula is due to Woolhouse.

Replacing the derivatives by their values in terms of differences we have Lubbock's Formula,[*]

$$(3) \qquad f(a)+f\left(a+\frac{\omega}{m}\right)+f\left(a+\frac{2\omega}{m}\right)+\ldots\ldots+f(a+n\omega)$$

$$=m\left\{f(a)+f(a+\omega)+\ldots\ldots+f(a+n\omega)\right\}-\frac{m-1}{2}\left\{f(a+n\omega)+f(a)\right\}$$

$$-\frac{m^2-1}{12m}\left\{\Delta f(a+n\omega)-\Delta f(a)\right\}+\frac{m^2-1}{24m}\left\{\Delta^2 f(a+n\omega)-\Delta^2 f(a)\right\}$$

$$-\frac{(m^2-1)\,(19m^2-1)}{720m^3}\left\{\Delta^3 f(a+n\omega)-\Delta^3 f(a)\right\}+\ldots\ldots$$

Formula (3) involves advancing differences of $f(a+n\omega)$. If backward differences are employed we get a slightly modified result due to De Morgan,[†]

$$(4) \qquad f(a)+f\left(a+\frac{\omega}{m}\right)+f\left(a+\frac{2\omega}{m}\right)+\ldots\ldots+f(a+n\omega)$$

$$=m\left\{f(a)+f(a+\omega)+\ldots\ldots+f(a+n\omega)\right\}-\frac{m-1}{2}\left\{f(a+n\omega)+f(a)\right\}$$

$$-\frac{m^2-1}{12m}\left\{\Delta f(a+\overline{n-1}\,\omega)-\Delta f(a)\right\}$$

$$-\frac{m^2-1}{24m}\left\{\Delta^2 f(a+\overline{n-2}\,\omega)+\Delta^2 f(a)\right\}$$

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[*] **J. W. Lubbock**—*Camb. Phil. Trans.*, 3 (1829), p. 323.

[†] **De Morgan**—*Diff. Int. Calc.*, pp. 317-18, §191. It may be readily obtained by the use of symbolic operators; cf. **T. B. Sprague**—*J. I. A.*, 18 (1874), p. 309.

$$-\frac{(m^2-1)(19m^2-1)}{720m^3}\left\{\Delta^3 f(a+\overline{n-3}\omega)-\Delta^3 f(a)\right\}$$

$$-\frac{(m^2-1)(9m^2-1)}{480m^3}\left\{\Delta^4 f(a+\overline{n-4}\omega)+\Delta^4 f(a)\right\}-\dots\dots$$

These formulæ enable us to obtain the sum of a large number of equidistant terms from a smaller number of terms taken at intervals m times greater.

Example 1. Find the sum of

$$10^3+11^3+12^3+\dots\dots+50^3.$$

Here $a=10$ and $f(x)=x^3$. Let $\omega=10$ and $m=10$, then forming the difference table, we have

	Δ	Δ^2	Δ^3	Δ^4
$10^3=1000$				
	7000			
$20^3=8000$		12000		
	19000		6000	
$30^3=27000$		18000		0
	37000		6000	
$40^3=64000$		24000		
	61000			
$50^3=125000$				

Hence from Lubbock's formula §4.20(4)

$$\sum_{10}^{50} x^3=10\left\{f(10)+f(20)+f(30)+f(40)+f(50)\right\}-\frac{9}{2}\left\{f(50)+f(10)\right\}$$

$$-\frac{99}{120}\left\{\Delta f(40)-\Delta f(10)\right\}-\frac{99}{240}\left\{\Delta^2 f(30)+\Delta^2 f(10)\right\}$$

$$-\frac{99\times 1899}{720000}\left\{\Delta^3 f(20)-\Delta^3 f(10)\right\}$$

$$=2250000-567000-44550-14850$$

$$=1623600.$$

EXAMPLES

1. Find a rational and integral function of x of as low a degree as possible that shall assume the values 3, 12, 15 and -21 when x is equal to 3, 2, 1 and -1 respectively.

[*C. U., M.A. & M.Sc., 1936*]

2. Given the table of values

x	-3	-2	-1	0	1
y	16	7	4	1	-8

Find by means of Gregory-Newton formula an expression for y as a function of x.

[C. U., M.A. & M.Sc., 1934]

3. Find an expression for $f(x)$ from the following table:

x	1,	2,	3,	4
$f(x)$	8,	7,	6,	5

[C. U., M.A. & M.Sc., 1939]

4. Find an expression for $f(x)$ from the following table:

x	0	1	2	3	4	5
$f(x)$	-66	0	0	0	354	1824

Hence obtain $f(\frac{1}{2})$ to the nearest whole number.

5. Find an expression for $f(x)$ from the following table:

x	2	6	4	8	7
$f(x)$	5	205	57	497	330

6. Complete the series for $f(x)$ from $x=0$ to $x=12$ from the following data:

x	0	4	8	12
$f(x)$	2·714	2·884	3·037	3·175

7. Find the value of $f(50)$ from the given table:

x	40	48	56	64
$f(x)$	14·27	15·81	17·72	19·96

8. Correct the errors in the following table by the method of differences:

x	$f(x)=\frac{1}{x}$	x	$f(x)$
0·21	4·762	·31	3·226
·23	4·348	·33	3·030
·25	4·000	·35	2·857
·27	3·704	·37	2·703
·29	3·465	·39	2·564

9. The following table giving the values of the moon's altitude at equal intervals of time, contains an error in one and only one entry:

339°	36′	53″·6	360° + 3°	34′	27″·5
345	32	44 ·3	9	41	58 ·6
351	30	30 ·9	15	54	6 ·3
357	30	41 ·8	22	11	30 ·5

Utilize the theory of finite differences to detect the position of the error and correct it.

[*C. U., M.A. & M.Sc., 1935*]

10. Correct the error in the following ephemeris of Moon's latitude:

Date	Moon's lat.						
May 8·5	−1°	59′	54″·2	11·0	1	10	23·4
9·0	−1	22	44 ·2	11·5	1	46	12·4
9·5	−0	44	27 ·0	12·0	2	20	14·7
10·0	−0	5	45 ·3	12·5	2	51	51·2
10·5	+0	32	39 ·9				

11. Correct the error in the following table of Mean refraction corresponding to the apparent altitude of the star:

alt.	m.r.				
10°	5′	19″·2	20	2	38 ·8
12	4	27 ·5	22	2	23 ·3
14	3	49 ·5	24	2	10 ·2
16	3	18 ·4	26	1	58 ·9
18°	2′	57″·5			

12. Correct the errors in the following tabulation of

$$f(x) = 2x^3 - 25x - 40:$$

x	−3,	−2,	−1,	0,	1,	2,	3,	4,	5,	6,	7,	8,
$f(x)$	−19,	−6,	−17,	−40,	−63,	−79,	−61,	−4,	85,	242,	471,	784.

13. Compute the value log sin 9°22′ from the following table:

θ	log sin θ	θ	log sin θ
6°	9·01923	10°	9·23967
7°	9·08589	11°	9·28060
8°	9·14356	12°	9·31788
9°	9·19433		

14. Calculate the value of sin 33° 13′ 30″ from the following table :

θ	30°	31°	32°	33°	34°
sin θ	·5	·515	·5299	·5446	·5592

15. Obtain the annual premiums for ages 31 to 39 for an assurance of £100 from the following table :

Age	£	s	d	Age	£	s	d
20	1	19	6	40	3	4	10
25	2	3	11	45	3	15	8
30	2	9	5	50	4	10	7
35	2	16	2				

16. The following annual premiums for policies maturing at quinquennial ages being given, find the premiums for policies maturing at all ages from 45 to 50 :

Age	45	50	55	60	65
Premium	2·871	2·404	2·083	1·862	1·712

17. If $f(x)$ be of the form $a+bx+cx^2$, shew that

$$\int_0^1 f(x)dx=\frac{1}{12}\left\{5f(0)+8f(1)-f(2)\right\}.$$

18. Shew that $\displaystyle\int_{1\cdot5}^{2\cdot5} f(x)dx=\frac{1}{24}\left\{f(1)+22f(2)+f(3)\right\}$.

19. Obtain the approximate formula

$$\int_{-1}^{1} f(x)dx=\frac{13\{f(1)+f(-1)\}-\{f(3)+f(-3)\}}{12}$$

shewing up to what order of differences it holds.

20. Prove the approximate formula

$$\int_0^{10} f(x)dx=2\cdot5\{f(1)+f(4)+f(6)+f(9)\},$$

and show that the formula involves a small second difference error.

21. If $f(x)=a+bx+cx^2$, shew that

$$\int_1^3 f(x)dx=2f(2)+\frac{1}{12}\left\{f(0)-2f(2)+f(4)\right\}.$$

Hence find an approximate value of

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\frac{x^2}{10}}\,dx.$$

22. Prove that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)dx = \frac{1}{2}\left\{f\left(-\frac{1}{2}\right) + f\left(\frac{1}{2}\right)\right\} + \frac{1}{24}\left\{\Delta f\left(-\frac{3}{2}\right) - \Delta f\left(\frac{1}{2}\right)\right\}$$

approximately.

Hence find the value of $\int_1^3 f(x)dx$, having given

$f(0)=105$, $f(1)=212$, $f(2)=421$, $f(3)=749$ and $f(4)=1050$.

23. Establish Shovelton's integration formula

$$\int_0^{10} f(x)dx = \frac{5}{126}\Big[8\{f(0)+f(10)\} + 35\{f(1)+f(3)+f(7)+f(9)\}$$
$$+ 15\{f(2)+f(4)+f(6)+f(8)\} + 36f(5)\Big].$$

24. Find the values of $\int_0^6 f(x)dx$ from the following tables:

	x	0	1	2	3	4	5	6
(*i*)	$f(x)$	·146	·161	·176	·190	·204	·217	·230
(*ii*)	$f(x)$	1	·96154	·92456	·88900	·85480	·82193	·79031

25. Find by Weddle's formula the values of

(*i*) $\int_0^1 \frac{dx}{1+x^2}$.   (*ii*) $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$.

26. Find by (1) Weddle's and (2) Hardy's formulæ the value of

$$\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1+x^2}}.$$

27. Find the value of $\int_{20^\circ}^{28^\circ} \cos\theta\, d\theta$ from the table at p. 105.

28. Use an approximate formula to evaluate

$$\int_{100}^{130} (x^3+x^2+1)dx.$$

29. Find by Lubbock's method the value of $\sum_{100}^{200} \frac{1}{x^2}$.

30. Find by Euler-Maclaurin formula the values of

(*i*) $\frac{1}{201^2}+\frac{1}{203^2}+\frac{1}{205^2}+\dots\dots+\frac{1}{299^2}$.

(*ii*) $\frac{1}{101^2}+\frac{1}{103^2}+\frac{1}{105^2}+\dots\dots$ *ad infinitum*.

31. Calculate by (i) Lubbock's formula, (*ii*) Euler-Maclaurin formula

$$\sum_{50}^{100}\frac{1}{x^2},\quad \sum_{100}^{150}\frac{1}{x^2}.$$

32. Calculate by (*i*) Euler-Maclaurin formula, (*ii*) Gregory's formula

$$\int_{25}^{30}\frac{dx}{x},\quad \int_{100}^{105}\frac{dx}{x}.$$

33. Prove that Gregory's formula may be obtained by making m increase indefinitely in Lubbock's formula.

34. Calculate $\int_{1\cdot05}^{1\cdot10} f(x)dx$ from the following table:

x	1·05	1·06	1·07	1·08	1·09	1·10
$f(x)$	1·25386	1·26996	1·28619	1·30254	1·31903	1·33565.

35. A series consists of nine terms. The sum of the first and the last term is 73·84 and the sums of other pairs of terms equidistant from the first and the last are respectively 73·07, 72·52 and 72·19. Find the middle term.

36. The number of deaths at age 44 is calculated from a mortality table giving the number of deaths at quinquennial age points 32, 37, 42, 47 and 52, and is estimated at 1000. Calculate what must have been the tabular number of deaths at age 32, it being given that the number of deaths at ages 37, 42, 47 and 52 were respectively 869, 952, 1112 and 1371.

37. Prove that Weddle's rule for approximate integration can be obtained by the combination of a 7-ordinate Simpson's formula with a 7-ordinate Three-eighths formula in the ratio of 9 : −4.

38. The numbers of members in a friendly society were available for the following years:

1922	1923	1924	1925	1928
995	998	1003	996	976.

The numbers for the years 1926 and 1927 were calculated on the assumption of a constant fourth difference. Subsequently the actual number for the year 1926 was discovered to be 1002, and a fresh estimate for 1927 was made. Find the original estimates for the years 1926 and 1927 and the revised one for 1927.

39. The following table furnishes, at noon on the given dates, the value of $\log D$, where D is the distance of Venus from the Earth measured in radii of the earth's orbit:

	$\log D$
July 1, 1912	·2393732
2	·2394118
3	·2394358
5	·2394402
6	·2394204
7	·2393860

Determine as accurately as you can from the given data—

(*i*) the value of $\log D$ at noon on July 4;

(*ii*) the time at which $\log D$ was greatest, and the corresponding value of $\log D$.

[*C. U., M.A. & M.Sc., 1934*]

40. From the following table of expectation of life, estimate E_{28}:

x	20	25	30	35
E_x	44·21	40·00	35·81	31·71.

If in addition the values for ages 10(53·08) and 15 (48·57) are given, obtain a revised estimate for E_{28}.

CHAPTER V

DIVIDED DIFFERENCES

5.1. Divided Differences. In previous chapters we have taken the argument x to increase by some constant quantity so that the difference between any two successive values remain the same. Here we shall consider the case when the argument increases by unequal increments, so that the difference between two successive values are different. The values of the argument are here unequally spaced. We introduce here another class of differences which goes by the name of *Divided Differences.* *

Take a function $f(x)$ whose values are given for $x=x_0, x_1, x_2, \ldots\ldots, x_n$. The intervals x_1-x_0, x_2-x_1, x_3-x_2, etc., need not be equal. The divided difference of $f(x)$ for the arguments x_0, x_1 is denoted by $f(x_0x_1)$, or by the symbol $[x_0x_1]$, and is defined by the relation

$$(1) \qquad f(x_0x_1)=\frac{f(x_0)-f(x_1)}{x_0-x_1}=\frac{f(x_1)-f(x_0)}{x_1-x_0}=f(x_1x_0).$$

Similarly $$f(x_1x_2)=\frac{f(x_1)-f(x_2)}{x_1-x_2}, \text{ and so on.}$$

These divided differences with two arguments are called *Divided Differences of the first order.*

Two divided differences of the first order having a common argument defines a *Divided Difference of the second order* involving three arguments. Thus

$$(2) \qquad f(x_0x_1x_2)=[x_0x_1x_2]=\frac{f(x_0x_1)-f(x_1x_2)}{x_0-x_2}.$$

Similarly a *Divided Difference of the third order* involving four arguments is defined by

$$(3) \qquad f(x_0x_1x_2x_3)=[x_0x_1x_2x_3]=\frac{f(x_0x_1x_2)-f(x_1x_2x_3)}{x_0-x_3}.$$

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* This term was first used by **De Morgan**, *Diff. and Int. Calc.* (1842), p. 550, and afterwards by **Oppermann**, *Journ. Inst. Act.*, 15 (1869), p. 146.

In general, a *Divided Difference of the nth order* involves $(n+1)$ arguments and is defined by

$$(4)\qquad f(x_0x_1x_2\ldots x_n)=[x_0x_1x_2\ldots x_n]$$
$$=\frac{f(x_0x_1\ldots x_{n-1})-f(x_1x_2\ldots x_n)}{x_0-x_n}.$$

We can exhibit these results in a table of divided differences as follows:

x_0	$f(x_0)$				
		$f(x_0x_1)$			
x_1	$f(x_1)$		$f(x_0x_1x_2)$		
		$f(x_1x_2)$		$f(x_0x_1x_2x_3)$	
x_2	$f(x_2)$		$f(x_1x_2x_3)$		$f(x_0x_1x_2x_3x_4)$.
		$f(x_2x_3)$		$f(x_1x_2x_3x_4)$	
x_3	$f(x_3)$		$f(x_2x_3x_4)$		
		$f(x_3x_4)$.			
x_4	$f(x_4)$				

Example. Calculate the divided differences of $f(x)$ from the following table:

x	1,	2,	4,	7,	12
$f(x)$	22,	30,	82,	106,	206.

[*C. U. M.A. & M.Sc., 1938*]

The divided differences are exhibited in the following table:

x	$f(x)$				
1	22				
		8			
2	30		6		
		26		$-1\cdot6$	
4	82		$-3\cdot6$		$\cdot192\ldots$
		8		$\cdot51$	
7	106		$1\cdot5$		
		20			
12	206				

5.2. Relation between Divided Differences and Ordinary Differences. Taking the interval of the argument x to be constant, equal to ω, we have

$$(1)\qquad f(x_0x_1)=\frac{f(x_1)-f(x_0)}{x_1-x_0}=\frac{f(x_0+\omega)+f(x_0)}{\omega}=\underset{\omega}{\Delta}f(x_0).$$

If in (1) we put $\omega=1$, we get

$$f(x_0x_1)=\Delta f(x_0).$$

Similarly
$$f(x_0x_1x_2)=\frac{f(x_0x_1)-f(x_1x_2)}{x_0-x_2}$$
$$=\frac{\underset{\omega}{\Delta}f(x_0)-\underset{\omega}{\Delta}f(x_1)}{x_0-x_2}$$
$$=\frac{1}{2}\underset{\omega}{\Delta}^2f(x_0).$$

When $\omega=1$, we have
$$f(x_0x_1x_2)=\frac{1}{2}\Delta^2f(x_0).$$

Again
$$f(x_0x_1x_2x_3)=\frac{f(x_0x_1x_2)-f(x_1x_2x_3)}{x_0-x_3}$$
$$=\frac{1}{2}\,\frac{\underset{\omega}{\Delta}^2f(x_0)-\underset{\omega}{\Delta}^2f(x_1)}{x_0-x_3}$$
$$=\frac{1}{3\,!}\underset{\omega}{\Delta}^3f(x_0).$$
$$=\frac{1}{3\,!}\Delta^3f(x_0),\text{ when }\omega=1.$$

In general, assume

(2) $$f(x_0,\ x_0+\omega,\ x_0+2\omega,\ldots,x_0+n\omega)=\frac{1}{n\,!}\underset{\omega}{\Delta}^n f(x_0),$$

then $f(x_0,\ x_0+\omega,\ x_0+2\omega,\ldots x_0+\overline{n+1}\ \omega)$
$$=\frac{f(x_0,\ x_0+\omega,\ldots,x_0+n\omega)-f(x_0+\omega,\ x_0+2\omega,\ldots,x_0+n\omega+\omega)}{x_0-(x_0+n\omega+\omega)}$$

(3) $$=\frac{1}{n\,!}\left\{\underset{\omega}{\Delta}^n f(x_0)-\underset{\omega}{\Delta}^n f(x_0+\omega)\right\}\cdot\frac{1}{-(n+1)\omega}=\frac{1}{(n+1)\,!}\underset{\omega}{\Delta}^{n+1}f(x_0).$$

Now (2) holds for $n=1, 2, 3$ and hence from (3) it will hold for $n=4, 5, 6,\ldots\ldots$ Thus (2) holds generally.

Again putting $\omega=1$, x for x_0, we have

(4) $$f(x,\ x+1,\ x+2,\ldots,x+n)=\frac{1}{n\,!}\Delta^n f(x).$$

Thus divided differences are generalisations of ordinary differences with constant increment of the argument.

A direct proof of (2) depends on the properties of §5.8, *q.v.*

5.3. Divided Differences are Symmetric Functions of the Arguments.

We have by definition

$$f(x_0x_1)=\frac{f(x_0)}{x_0-x_1}+\frac{f(x_1)}{x_1-x_0}.$$

Again

$$f(x_0x_1x_2)=\frac{f(x_0x_1)-f(x_1x_2)}{x_0-x_2}.$$

$$=\frac{1}{x_0-x_2}\left\{\frac{f(x_0)}{x_0-x_1}+\frac{f(x_1)}{x_1-x_0}\right\}+\frac{1}{x_2-x_0}\left\{\frac{f(x_1)}{x_1-x_2}+\frac{f(x_2)}{x_2-x_1}\right\}$$

$$(1)=\frac{f(x_0)}{(x_0-x_1)(x_0-x_2)}+\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)}+\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}.$$

In general, it can be easily proved that a divided difference of the nth order

$$(2)\qquad f(x_0x_1x_2\ldots x_n)=\frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}$$

$$+\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}+\ldots+\frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}.$$

It is easily seen that *the value of the divided difference is unaltered by an interchange of the arguments in any order*, and thus the divided difference is a symmetric function of its arguments.

When the interval is constant, equal to ω, we have

$$f(x_0,\ x_0+\omega,\ x_0+2\omega,\ldots,\ x_0+n\omega)$$

$$=\frac{f(x_0+n\omega)}{n!\,\omega^n}-\frac{f(x_0+\overline{n-1}\omega)}{(n-1)!\,1!\,\omega^n}+\frac{f(x_0+\overline{n-2}\omega)}{(n-2)!\,2!\,\omega^n}-\ldots+(-1)^n\frac{f(x_0)}{n!\,\omega^n}$$

$$=\frac{1}{n!}\mathop{\Delta^n}_{\omega} f(x_0).$$

5.4. Properties of Divided Differences.

(i) *If $f(x)=\phi(x)+\psi(x)$ numerically for a set of values of the argument x, then any divided difference of $f(x)$ formed from those values is equal to the sum of the corresponding divided differences of $\phi(x)$ and $\psi(x)$.*

For, $$f(x_0x_1)=\frac{f(x_0)-f(x_1)}{x_0-x_1}=\frac{\phi(x_0)-\phi(x_1)+\psi(x_0)-\psi(x_1)}{x_0-x_1}$$
$$=\phi(x_0x_1)+\psi(x_0x_1);$$
and similarly for differences of higher order.

(ii) *A divided difference of $c.f(x)$, where c is constant, is c times the corresponding divided difference of $f(x)$.*

For example $$\frac{cf(x_0)-cf(x_1)}{x_0-x_1}=c\,f(x_0x_1).$$

(iii) *The divided differences of the nth order of x^n are constant.*

Here $f(x)=x^n$.

$$\therefore\quad f(x_0x_1)=\frac{x_0^n-x_1^n}{x_0-x_1}=x_0^{n-1}+x_1x_0^{n-2}+\ldots+x_1^{n-1},$$
a homogeneous function of x_0, x_1 of degree $n-1$. Again
$$f(x_0x_1x_2)=\frac{x_0^n}{(x_0-x_1)(x_0-x_2)}+\frac{x_1^n}{(x_1-x_0)(x_1-x_2)}+\frac{x_2^n}{(x_2-x_0)(x_2-x_1)}$$
$$=\frac{1}{x_0-x_2}\left\{\frac{x_0^n-x_1^n}{x_0-x_1}-\frac{x_1^n-x_2^n}{x_1-x_2}\right\}$$
$$=\frac{x_0^{n-1}+x_1x_0^{n-2}+\ldots+x_1^{n-1}}{x_0-x_2}-\frac{x_1^{n-1}+x_2x_1^{n-2}+\ldots+x_2^{n-1}}{x_0-x_2}$$
$$=x_0^{n-2}+x_2x_0^{n-3}+\ldots+x_2^{n-2}+x_1(x_0^{n-3}+x_2x_0^{n-4}+\ldots+x_2^{n-3})+\ldots$$
which is a homogeneous function of x_0, x_1, x_2 of degree $n-2$.

Thus the operation of taking the divided difference lowers the degree by unity.

In general
$$f(x_0x_1x_2\ldots x_p)=\sum_{r=0}^{p}\frac{x_r^n}{(x_r-x_0)\,(x_r-x_1)\ldots{}^*(x_r-x_p)}$$
is a homogeneous function of x_0, $x_1,\ldots,x_p$ of degree $n-p$, the * indicating that the factor x_r-x_r has been omited. If $p=n$, we see that the divided difference of $f(x_0x_1\ldots x_n)$ is a constant.

The divided differences of order $(n+1)$ of x^n are zero.

(iv) *The divided differences of order n of a polynomial of the nth degree are constant.*

The divided differences of order n of each of the terms whose degree is less than n is zero, and of the term whose degree is n is constant.

5.5. Divided Difference expressed by Determinants.

We have

$$f(x_0x_1x_2\ldots x_n)$$

$$= \sum_{r=0}^{n} \frac{f(x_r)}{(x_r-x_0)(x_r-x_1)\ldots(x_r-x_{r-1})(x_r-x_{r+1})\ldots(x_r-x_n)}$$

$$= \sum_{r=0}^{n} \frac{f(x_r)\ \times \text{difference product of } x_0,x_1,\ldots,x_{r-1},\,*\,x_{r+1},\ldots,x_n}{\text{difference product of } x_0,x_1,\ldots,x_r,\ldots,x_n}$$

the $*$ indicating that x_r has been omitted from the arguments.

Now, by Cauchy's generalisation of Vandermonde's theorem, we have the difference product of $x_0, x_1,\ldots x_r,\ldots,x_n = \Pi_{j>i}(x_j-x_i)$

$$= \begin{vmatrix} 1 & 1 & 1 \ldots\ldots & 1 \\ x_0 & x_1 & x_2 \ldots\ldots & x_n \\ x_0^2 & x_1^2 & x_2^2 \ldots\ldots & x_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ x_0^n & x_1^n & x_2^n \ldots\ldots & x_n^n \end{vmatrix} \tag{1}$$

Also $\sum_{r=0}^{n} f(x_r) \times$ difference product of $x_0, x_1,\ldots,x_{r-1},x_{r+1}\ldots,x_n$

$$= \begin{vmatrix} 1 & 1 & 1 \ldots\ldots & 1 \\ x_0 & x_1 & x_2 \ldots\ldots & x_n \\ x_0^2 & x_1^2 & x_2^2 \ldots\ldots & x_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ x_0^{n-1} & x_1^{n-1} & x_2^{n-1} \ldots & x_n^{n-1} \\ f(x_0) & f(x_1) & f(x_2) \ldots & f(x_n) \end{vmatrix}$$

Hence

$$f(x_0x_1x_2\ldots x_n) = \begin{vmatrix} 1 & 1 & 1 & \ldots 1 \\ x_0 & x_1 & x_2 & \ldots x_n \\ x_0^2 & x_1^2 & x_2^2 & \ldots x_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ x_0^{n-1} & x_1^{n-1} & x_2^{n-1} & \ldots x_n^{n-1} \\ f(x_0) & f(x_1) & f(x_2) & \ldots f(x_n) \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 & \ldots 1 \\ x_0 & x_1 & x_2 & \ldots x_n \\ x_0^2 & x_1^2 & x_2^2 & \ldots x_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ x_0^{n-1} & x_1^{n-1} & x_2^{n-1} & \ldots x_n^{n-1} \\ x_0^n & x_1^n & x_2^n & \ldots x_n^n \end{vmatrix} \tag{2}$$

Thus divided difference of order n is expressed as the quotient of two determinants each of order $(n+1)$.

5.6. Newton's Interpolation Formula with Divided Differences. Writing x for x_0 in §5.1(4) we have

$$f(xx_1x_2\ldots x_n)=\frac{f(xx_1x_2\ldots x_{n-1})-f(x_1x_2\ldots x_n)}{x-x_n}.$$

Hence

$$f(xx_1x_2\ldots x_{n-1})=f(x_1x_2\ldots x_n)+(x-x_n)f(xx_1x_2\ldots x_n).$$

Similarly

$$f(xx_1x_2\ldots x_{n-2})=f(x_1x_2\ldots x_{n-1})+(x-x_{n-1})f(xx_1x_2\ldots x_{n-1}),$$

$$f(xx_1x_2\ldots x_{n-3})=f(x_1x_2\ldots x_{n-2})+(x-x_{n-2})f(xx_1x_2\ldots x_{n-2}),$$

....

$$f(xx_1)=f(x_1x_2)+(x-x_2)f(xx_1x_2),$$

$$f(x)=f(x_1)+(x-x_1)f(xx_1).$$

Therefore

$$\begin{aligned}f(x)&=f(x_1)+(x-x_1)f(xx_1)\\&=f(x_1)+(x-x_1)f(x_1x_2)+(x-x_1)(x-x_2)f(xx_1x_2)\\&=f(x_1)+(x-x_1)f(x_1x_2)+(x-x_1)(x-x_2)f(x_1x_2x_3)\\&\qquad\qquad+(x-x_1)(x-x_2)(x-x_3)f(xx_1x_2x_3).\end{aligned}$$

Proceeding in this way we ultimately have

$$\begin{aligned}f(x)=f(x_1)&+(x-x_1)f(x_1x_2)+(x-x_1)(x-x_2)f(x_1x_2x_3)+\ldots\\&+(x-x_1)(x-x_2)\ldots(x-x_{r-1})f(x_1x_2\ldots x_r)+\ldots\\&+(x-x_1)(x-x_2)\ldots(x-x_{n-1})f(x_1x_2\ldots x_n)\\&+(x-x_1)(x-x_2)\ldots(x-x_n)f(xx_1x_2\ldots x_n).\end{aligned}\tag{1}$$

This can be put as

$$f(x)=f(x_1)+\sum_{r=1}^{n-1}(x-x_1)(x-x_2)\ldots(x-x_r)f(x_1x_2\ldots x_{r+1})+R_n(x),\tag{2}$$

where

$$R_n(x)=(x-x_1)(x-x_2)\ldots(x-x_n)f(xx_1x_2\ldots x_n).\tag{3}$$

This is Newton's general interpolation formula* with the remainder term $R_n(x)$, and no restriction is imposed on the form of the function $f(x)$. This can be considered as the *Basic Formula* in Finite Differences.

When the values of $f(x)$ for $x=x_1, x_2, \ldots, x_n$ are known, the evaluation of $f(x)$ depends on the value of $R_n(x)$. If $R_n(x)$ be known or negligible, the value of $f(x)$ is obtained from (2).

---

* Newton—*Principia* (1687), Book III, Lemma v.

If $f(x)$ be a polynomial of degree $n-1$, the divided difference of order $(n-1)$ is constant and of order n is zero, *i.e.*,

$$f(xx_1x_2\ldots x_n)=0,$$

so that in this case $$R_n(x)\equiv 0;$$
and

$$f(x)=f(x_1)+\sum_{r=1}^{n-1}(x-x_1)(x-x_2)\ldots(x-x_r)f(x_1x_2\ldots x_{r+1}). \tag{4}$$

If $f(x)$ be not a polynomial, we see that

$$R_n(x)=0, \quad \text{for} \quad x=x_1, x_2, \ldots, x_n,$$

so that the right-hand side of (4) gives a polynomial of degree $(n-1)$ as the equivalent of $f(x)$ for the given set of arguments.

A polynomial, of degree $n-1$ at most, whose values at $x_1, x_2, \ldots, x_n$ are the same as the values of $f(x)$ at these points, is called an **Interpolation Polynomial** of $f(x)$ and is denoted by $I_{n-1}(x)$. To obtain the interpolation formula (2) we have to add $R_n(x)$ to it. Thus

$$f(x)=I_{n-1}(x)+R_n(x)$$

where $I_{n-1}(x)$ stands for $f(x)$ in (4).

An interpolation polynomial is fixed by the values of the function at the given points. It does not depend on the order in which these points are considered.

Example 1. Find the interpolation polynomial from the following table:

x	0	1	2	5
$f(x)$	2	3	12	147.

The difference table is exhibited below:

x	$f(x)$			
0	2			
		1		
1	3		4	
		9		1
2	12		9	
		45		
5	147			

Hence

$$\begin{aligned} f(x)&=f(0)+xf(0,1)+x(x-1)f(0,1,2)+x(x-1)(x-2)f(0,1,2,5)\\ &=2+x+4x(x-1)+x(x-1)(x-2)\\ &=x^3+x^2-x+2. \end{aligned}$$

Example 2. Find the expression for $f(x)$ from the following data:

x	1	2	4	5	7	9
$f(x)$	-96	0	120	0	-780	0.

Since $f(x)=0$ for $x=2$, 5 and 9, it must be of the form

$$f(x)=(x-2)(x-5)(x-9)\phi(x),$$

where $\phi(x)$ is a polynomial of the second degree in x.

$$\therefore\ f(1)=-1.-4.-8\phi(1)=-32\phi(1), \text{ or } \phi(1)=3$$

$$f(4)=\quad 2.-1.-5\phi(4)=10\phi(4), \quad \text{or } \phi(4)=12$$

$$f(7)=\quad 5.\ 2.-2\phi(7)=-20\phi(7), \text{ or } \phi(7)=39$$

$$\begin{array}{ll} 3 & \\ & 1 \\ 9 & \end{array}$$

Hence $\phi(x)=3+(x-1)3+(x-1)(x-4)=x^2-2x+4.$

$$\therefore\ f(x)=(x-2)(x-5)(x-9)(x^2-2x+4).$$

5.7. The Remainder Term. In §5.6(2) put

$$P_{n-1}(x)=f(x_1)+\sum_{r=1}^{n-1}(x-x_1)(x-x_2)\ldots(x-x_r)f(x_1x_2\ldots x_{r+1}).$$

Then $$R_n(x)=f(x)-P_{n-1}(x).$$

Evidently $P_{n-1}(x)$ is a polynomial of $(n-1)th$ degree and its $(n-1)$th derivative is

$$P_{n-1}^{(n-1)}(x)=(n-1)!f(x_1x_2\ldots x_n).$$

Let us now suppose that the function $f(t)$ of the real variable t and its first $n-1$ derivatives are finite and continuous, and $f^{(n)}(t)$ exists, in the interval (a, b) bounded by the greatest and least of the arguments

$$x, x_1, x_2, \ldots, x_n.$$

Then since $R_n(t)$ vanishes for $t=x_1, x_2, \ldots, x_n$, we know by Rolle's theorem that $R'_n(t)$ will vanish at $(n-1)$ points in (a, b); $R''_n(t)$ will vanish at $n-2$ points in (a, b). Ultimately

$$R_n^{(n-1)}(\eta)=0,$$

where η is some point in (a, b). Hence

$$f^{(n-1)}(\eta)-P_{n-1}^{(n-1)}(\eta)=0,$$

so that

$$f(x_1x_2\ldots x_n)=\frac{f^{(n-1)}(\eta)}{(n-1)!}. \tag{1}$$

Thus the divided difference of $(n-1)$th order is expressed in terms of the $(n-1)$th derivative of $f(x)$ at some point η in the interval (a, b). Hence

$$f(xx_1x_2\ldots x_n)=\frac{f^{(n)}(\xi)}{n\,!}, \tag{2}$$

where ξ is some point in (a, b). Therefore we have

$$R_n(x)=(x-x_1)(x-x_2)\ldots(x-x_n)\,\frac{f^{(n)}(\xi)}{n\,!}, \tag{3}$$

where ξ is some point in (a, b).

This enables us to find an upper limit of the error committed in omitting the remainder term.

5.8. Gregory-Newton Interpolation Formula—Forward Differences. Gregory-Newton formula is a special case of Newton's formula §5.6(1) when the arguments are equidistant. Let

$$x_r=a+(r-1)\omega,\quad r=1,2,3,\ldots n.$$

We have by §5.2(2) $f(x_1x_2x_3\ldots x_n)=\dfrac{1}{(n-1)\,!}\underset{\omega}{\Delta}^{n-1}f(x_1)$,

and by §5.7(2) $\qquad f(xx_1x_2\ldots x_n)=\dfrac{f^{(n)}(\xi)}{n\,!}$.

Hence substituting these values in Newton's Interpolation Formula §5.6(1), we have

$$f(x)=f(a)+(x-a)\underset{\omega}{\Delta}f(a)+\frac{(x-a)(x-a-\omega)}{2\,!}\underset{\omega}{\Delta}^2f(a)+\ldots \tag{1}$$

$$+\frac{(x-a)(x-a-\omega)(x-a-2\omega)\ldots(x-a-\overline{n-2}\,\omega)}{(n-1)\,!}\underset{\omega}{\Delta}^{n-1}f(a)+R_n(x),$$

where $\qquad R_n(x)=\dfrac{(x-a)(x-a-\omega)\ldots(x-a-\overline{n-1}\,\omega)}{n\,!}f^{(n)}(\xi)$,

and ξ lies in the interval bounded by the greatest and least of x, a, $a+(n-1)\omega$.

This is Newton's formula with forward differences. It was discovered by James Gregory in 1670 and is known as Gregory-Newton Formula.

It can be put in factorial notation thus:

$$(2) \qquad f(x)=f(a)+(x-a)\underset{\omega}{\Delta}f(a)+\frac{(x-a)^{(2\omega)}}{2!}\underset{\omega}{\Delta^2}f(a)+\dots$$

$$+\frac{(x-a)^{(\overline{n-1}\,\omega)}}{(n-1)!}\underset{\omega}{\Delta^{n-1}}f(a)+\frac{(x-a)^{(n\omega)}}{n!}f^{(n)}(\xi);$$

or in difference notation this:

$$(3) \quad f(x)=f(a)+(x-a)\frac{\Delta f(a)}{\omega}+\frac{(x-a)^{(2\omega)}}{2!}\frac{\Delta^2 f(a)}{\omega^2}+\dots$$

$$+\frac{(x-a)^{(\overline{n-1}\,\omega)}}{(n-1)!}\frac{\Delta^{n-1}f(a)}{\omega^{n-1}}+\frac{(x-a)^{(n\omega)}}{n!}f^{(n)}(\xi).$$

It can be easily seen that the differences employed lie on a line sloping downwards from $f(a)$. See table of §1.2.

When $\omega \longrightarrow 0$, we obtain Taylor's theorem

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\dots$$

$$+\frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{(x-a)^n}{n!}f^n(\xi),$$

where ξ lies in the interval (a, x).

Putting $a+x\omega$ in place of x, the formula (1) can be put in the simplified form

$$(4) \quad f(a+x\omega)=f(a)+x\Delta f(a)+\frac{x(x-1)}{2!}\Delta^2 f(a)+\dots$$

$$+\frac{x(x-1)\dots(x-n+2)}{(n-1)!}\Delta^{n-1}f(a)+\frac{x(x-1)(x-2)\dots(x-n+1)}{n!}\omega^n f^{(n)}(\xi).$$

In particular,

$$(5) \quad f(a+n\omega)=f(a)+n\Delta f(a)+\frac{n(n-1)}{2!}\Delta^2 f(a)+\dots+n\Delta^{n-1}f(a)+\Delta^n f(a).$$

If $f(x)$ be a polynomial of degree $n-1$, and $a=0$ and $\omega=1$, we have

$$f(x)=f(0)+x\Delta f(0)+\frac{x(x-1)}{2!}\Delta^2 f(0)+\dots+\frac{x(x-1)\dots(x-n+2)}{(n-1)!}\Delta^{n-1}f(0).$$

5.9. Newton's Interpolation Formula—Backward Differences.

Let us construct a table of differences:

$$\begin{array}{lllll}
a-3\omega & f(a-3\omega) & & & \\
 & & \Delta f(a-3\omega) & & \\
a-2\omega & f(a-2\omega) & & \Delta^2 f(a-3\omega) & \\
 & & \Delta f(a-2\omega) & & \Delta^3 f(a-3\omega) \\
a-\omega & f(a-\omega) & & \Delta^2 f(a-2\omega) & \\
 & & \Delta f(a-\omega) & & \\
a & f(a) & & &
\end{array}$$

Let $x_r = a-(r-1)\omega, \quad r=1, 2, 3, \ldots n,$

then by §5.2 $f(x_1 x_2) = \underset{\omega}{\Delta} f(a-\omega),$

$$f(x_1 x_2 x_3) = \frac{1}{2\,!}\underset{\omega}{\Delta}^2 f(a-2\omega),$$

and generally $f(x_1 x_2 \ldots x_{r+1}) = \dfrac{1}{r\,!}\underset{\omega}{\Delta}^r f(a-r\omega).$

Thus we have

$$(1) \qquad f(x) = f(a) + (x-a)\underset{\omega}{\Delta} f(a-\omega) + \frac{(x-a)\,(x-a+\omega)}{2\,!}\underset{\omega}{\Delta}^2 f(a-2\omega) + \ldots$$

$$+ \frac{(x-a)\,(x-a+\omega)\ldots(x-a+\overline{n-2}\omega)}{(n-1)\,!}\underset{\omega}{\Delta}^{n-1} f(a-\overline{n-1}\omega)$$

$$+ \frac{(x-a)\,(x-a+\omega)\ldots\,(x-a+\overline{n-1}\omega)}{n\,!} f^{(n)}(\xi),$$

where ξ lies in the interval bounded by the greatest and the least of x, a, $a-(n-1)\omega$.

From (1) we easily get

$$(2) \qquad f(a+x\omega) = f(a) + x\Delta f(a-\omega) + \frac{x(x+1)}{2\,!}\Delta^2 f(a-2\omega) + \ldots$$

$$+ \frac{x(x+1)\,(x+2)\ldots(x+n-2)}{(n-1)\,!}\Delta^{n-1} f(a-\overline{n-1}\omega)$$

$$+ \frac{x(x+1)\,(x+2)\ldots(x+n-1)}{n\,!}\,\omega^n f^{(n)}(\xi),$$

and

$$(8) \qquad f(a-x\omega)=f(a)-x\Delta f(a-\omega)+\frac{x(x-1)}{2!}\Delta^2 f(a-2\omega)+\ldots$$

$$+(-1)^{n-1}\frac{x(x-1)\ldots(x-n+2)}{(n-1)!}\Delta^{n-1}f(a-\overline{n-1}\,\omega)$$

$$+(-1)^n\frac{x(x-1)\ldots(x-n+1)}{n!}\omega^n f^{(n)}(\xi).$$

The differences employed here lie on a line sloping upwards from $f(a)$. The corresponding interpolation polynomial is obtained by omitting the remainder term.

5.10. Lagrange's Interpolation Formula without Differences.

Let $f(x)$ be a function of the real variable x defined in the interval (a, b) to which belongs the set of arguments $x_1, x_2, \ldots, x_n$, and possessing derivatives. From §5.3(2) we have

$$f(xx_1x_2\ldots x_n)=\frac{f(x)}{(x-x_1)(x-x_2)\ldots(x-x_n)}+\frac{f(x_1)}{(x_1-x)(x_1-x_2)\ldots(x_1-x_n)}$$

$$+\ldots\ldots+\frac{f(x_n)}{(x_n-x)(x_n-x_1)\ldots(x_n-x_{n-1})}.$$

Hence

$$(1) \qquad f(x)=\frac{(x-x_2)(x-x_3)\ldots\ldots(x-x_n)}{(x_1-x_2)(x_1-x_3)\ldots\ldots(x_1-x_n)}f(x_1)$$

$$+\frac{(x-x_1)(x-x_3)\ldots\ldots(x-x_n)}{(x_2-x_1)(x_2-x_3)\ldots\ldots(x_2-x_n)}f(x_2)+\ldots$$

$$+\frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\ldots\ldots(x_n-x_{n-1})}f(x_n)$$

$$+(x-x_1)(x-x_2)\ldots\ldots(x-x_n)f(xx_1x_2\ldots\ldots x_n).$$

This is Lagrange's formula with the remainder term.

Putting $\qquad \phi(x)=(x-x_1)(x-x_2)\ldots(x-x_n)$,
we have (1) in the form

$$f(x)=\sum_{r=1}^{n}\frac{f(x_r)}{x-x_r}\cdot\frac{\phi(x)}{\phi'(x_r)}+R_n(x),$$

where
$$R_n(x) = \phi(x) f(xx_1x_2\ldots\ldots x_n)$$
$$= \phi(x)\frac{1}{n!}f^{(n)}(\xi), \quad \text{by §5.7(2)}$$

where ξ is some point in (a, b). Thus

$$(2) \qquad f(x) = \sum_{r=1}^{n} \frac{f(x_r)}{x - x_r} \cdot \frac{\phi(x)}{\phi'(x_r)} + \frac{1}{n!}\phi(x) f^{(n)}(\xi).$$

If $f(x)$ be an interpolation polynomial of degree $n-1$, its divided difference of nth order is zero, *i.e.*,

$$f(xx_1\ldots\ldots x_n) = 0$$

and the formula becomes

$$(3) \qquad f(x) = \sum_{r=1}^{n} \frac{f(x_r)}{x - x_r} \cdot \frac{\phi(x)}{\phi'(x_r)}.$$

5.11. Hermite's Expression of Divided Differences.

Hermite has given an expression of Divided Differences by definite integrals in the form

$$(1) \quad f(x_1x_2\ldots\ldots x_n) = \int_0^1 dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \ldots \int_0^{t_{n-2}} f^{(n-1)}(u_n)\, dt_{n-1},$$

where $u_n = (1-t_1)x_1 + (t_1 - t_2)x_2 + \ldots + (t_{n-2} - t_{n-1})x_{n-1} + t_{n-1}x_n$.

Evidently the independent variables t_1, t_2, $t_3, \ldots\ldots$ will all disappear when the definite integral is evaluated.

Suppose formula (1) is true for the n arguments $x_1, x_2, \ldots, x_n$ and let us proceed to the case of $n+1$ arguments with x_{n+1} and a new parameter t_n. Now

$$\int_0^{t_{n-1}} f^{(n)}(u_{n+1})\, dt_n = \int_0^{t_{n-1}} f^{(n)} \Big\{ (1-t_1)x_1 + (t_1 - t_2)x_2 + \ldots$$
$$+ (t_{n-1} - t_n)x_n + t_n x_{n+1} \Big\}\, dt_n$$

$$= \frac{f^{(n-1)}\{(1-t_1)x_1 + \ldots + (t_{n-2} - t_{n-1})x_{n-1} + t_{n-1}x_n\}}{x_n - x_{n+1}}$$

$$- \frac{f^{(n-1)}\{(1-t_1)x_1 + \ldots + (t_{n-2} - t_{n-1})x_{n-1} + t_{n-1}x_{n+1}\}}{x_n - x_{n+1}}.$$

Hence
$$\int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots\ldots \int_0^{t_{n-1}} f^{(n)}(u_{n+1})\,dt_n$$
$$=\frac{f(x_1x_2\ldots x_n)-f(x_1x_2\ldots x_{n-1}x_{n+1})}{x_n-x_{n+1}}$$

which can be put as
$$\frac{f(x_nx_1x_2\ldots x_{n-1})-f(x_1x_2\ldots x_{n-1}x_{n+1})}{x_n-x_{n+1}}$$
$$=f(x_nx_1x_2\ldots x_{n-1}x_{n+1})$$

(2)
$$=f(x_1x_2\ldots x_nx_{n+1}).$$

Thus if (1) holds for n, it will hold for $n+1$. Putting $n=2$ in (1) we have

$$\int_0^1 f'\left\{(1-t_1)x_1+t_1x_2\right\}dt_1=\frac{f(x_2)-f(x_1)}{x_2-x_1}=f(x_1x_2),$$

so that the formula holds for $n=2$. Hence by induction it holds for $n=3, 4, \ldots$, *i.e.*, it holds generally.

5.12. Expression of Divided Differences by Contour Integrals. Take an analytic function $f(t)$ of the complex variable t which is holomorphic in and along the contour C enclosing a simply connected region in which the points $z_1, z_2, \ldots, z_n$ are situated. Then by Cauchy's residue theorem

$$f(z)=\frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}\,dt.$$

Again the residue at $t=z_r$ of the function

$$\frac{f(t)}{(t-z_1)(t-z_2)\ldots(t-z_n)}$$

is
$$\frac{f(z_r)}{(z_r-z_1)(z_r-z_2)\ldots(z_r-z_{r-1})(z_r-z_{r+1})\ldots(z_r-z_n)}.$$

Hence

(1)
$$\frac{1}{2\pi i}\int_C \frac{f(t)}{(t-z_1)\,(t-z_2)\ldots(t-z_n)}\,dt$$
$$=\sum_{r=1}^{n}\frac{f(z_r)}{(z_r-z_1)\ldots.(z_r-z_{r-1})\,(z_r-z_{r+1})\ldots.(z_r-z_n)}$$
$$=f(z_1z_2\ldots z_n).$$

Thus the divided difference of order $n-1$ of $f(z)$ is expressed in the form of a contour integral.

5.13. Newton's Interpolation Formula in terms of Contour Integral.

Since
$$\frac{1}{t-z}=\frac{1}{t-z_1}+\frac{z-z_1}{t-z_1}\cdot\frac{1}{t-z},$$
$$\frac{1}{t-z}=\frac{1}{t-z_2}+\frac{z-z_2}{t-z_2}\,\frac{1}{t-z},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$
we have by repeated applications
$$\frac{1}{t-z}=\frac{1}{t-z_1}+\frac{z-z_1}{t-z_1}\cdot\frac{1}{t-z_2}+\frac{(z-z_1)(z-z_2)}{(t-z_1)(t-z_2)}\cdot\frac{1}{t-z_3}+\dots$$
$$+\frac{(z-z_1)(z-z_2)\dots(z-z_n)}{(t-z_1)(t-z_2)\dots(t-z_n)}\cdot\frac{1}{t-z}.$$
Hence
$$\frac{1}{2\pi i}\int_C\frac{f(t)}{t-z}\,dt$$
$$=\frac{1}{2\pi i}\int_C\frac{f(t)}{t-z_1}\,dt+\frac{1}{2\pi i}(z-z_1)\int_C\frac{f(t)}{(t-z_1)(t-z_2)}\,dt+\dots$$
$$+\frac{1}{2\pi i}(z-z_1)(z-z_2)\dots(z-z_n)\int_C\frac{f(t)}{(t-z_1)(t-z_2)\dots(t-z_n)(t-z)}\,dt.$$

Thus
$$f(z)=f(z_1)+(z-z_1)f(z_1z_2)+(z-z_1)(z-z_2)f(z_1z_2z_3)+\dots$$
$$+(z-z_1)(z-z_2)\dots(z-z_{n-1})f(z_1z_2\dots z_n)+R_n(z)$$
where
$$R_n(z)=\frac{1}{2\pi i}(z-z_1)(z-z_2)\dots(z-z_n)\int_C\frac{f(t)}{(t-z_1)\dots(t-z_n)\,(t-z)}\,dt.$$
which is Newton's general formula. Here $f(z)$ is not unrestricted as in §5.6 but is subject to the restrictions given in §5.12.

5.14. Confluent Divided Differences. In the definition of divided differences the arguments were all taken to be different from one another and it can be easily seen that if two or more of them become equal, the values of the divided differences become indeterminate. A limit may however be found of $f(x_1, x_1+\epsilon)$ as $\epsilon \longrightarrow 0$, and if this limit be finite, it can be taken as the value of $f(x_1 x_1)$. Such limiting values of divided differences with two or more equal arguments are called *Confluent Divided Differences.*

We have from §5.7

$$f(x_1 x_2 \ldots x_n) = \frac{f^{(n-1)}(\eta)}{(n-1)!},$$

where η lies in the interval bounded by the greatest and least of $x_1, x_2, \ldots, x_n$.

If all these arguments become equal to x_1, we have the confluent divided difference

$$f(x_1 x_1 \ldots x_1) = \frac{f^{(n-1)}(x_1)}{(n-1)!}. \tag{1}$$

Thus confluent divided differences can only be formed if the function $f(x)$ possesses the necessary derivatives.

For constructing a table, arguments having the same value are to be placed in a single group and (1) is to be put in place of the corresponding divided differences. The method is exhibited thus:

x_1	$f(x_1)$			
		$f'(x_1)$		
x_1	$f(x_1)$		$\frac{1}{2}f''(x_1)$	
		$f'(x_1)$		$f(x_1 x_1 x_1 x_2)$
x_1	$f(x_1)$		$f(x_1 x_1 x_2)$	
		$f(x_1 x_2)$		$f(x_1 x_1 x_2 x_2)$
x_2	$f(x_2)$		$f(x_1 x_2 x_2)$	
		$f'(x_2)$		$f(x_1 x_2 x_2 x_3)$
x_2	$f(x_2)$		$f(x_2 x_2 x_3)$	
		$f(x_2 x_3)$		$f(x_2 x_2 x_3 x_4)$
x_3	$f(x_3)$		$f(x_2 x_3 x_4)$	
		$f(x_3 x_4)$		
x_4	$f(x_4)$			

If all the arguments are equal to x_1, we easily get from Newton's formula Taylor's expansion

$$f(x) = f(x_1) + (x - x_1) f'(x_1) + \frac{(x-x_1)^2}{2!} f''(x_1) + \ldots$$

$$+ \frac{(x-x_1)^{n-1}}{(n-1)!} f^{(n-1)}(x_1) + R_n(x).$$

where
$$R_n(x)=\frac{(x-x_1)^n}{n!}f^{(n)}(\xi),$$

ξ being some point lying in the interval (x, x_1).

If there are k groups of equal arguments, p arguments equal to x_1, q arguments equal to x_2, r arguments equal to x_3,......, we have from §5.12(1)

$$f(x_1x_1\ldots x_2x_2\ldots x_3x_3\ldots)=\frac{1}{2\pi i}\int_C\frac{f(t)}{(t-x_1)^p(t-x_2)^q(t-x_3)^r\ldots}dt$$

$$=\frac{1}{(p-1)!}\frac{1}{(q-1)!}\frac{1}{(r-1)!}\cdots\frac{1}{2\pi i}\frac{\partial^{p-1}}{\partial x_1^{p-1}}\frac{\partial^{q-1}}{\partial x_2^{q-1}}\cdot\frac{\partial^{r-1}}{\partial x_3^{r-1}}\cdots$$

$$\int_C\frac{f(t)}{(t-x_1)(t-x_2)(t-x_3)\ldots}dt.$$

$$=\frac{1}{(p-1)!}\frac{1}{(q-1)!}\frac{1}{(r-1)!}\cdots\frac{\partial^{p+q+r+\cdots-k}}{\partial x_1^{p-1}\partial x_2^{q-1}\partial x_3^{r-1}\cdots\cdots}f(x_1x_2x_3\ldots).$$

Example 1. Find an expression for $f(x)$ in terms of powers of $(x-3)$ from the following data:

x	5	13	27	34	43
$f(x)$	43	1627	17335	35626	73687.

Constructing the table of differences and extending it to include repeated arguments for $x=3$, we obtain:

x	$f(x)$			
3	$f(3)=7$			
		$f'(3)=2$		
3	$f(3)=7$		$\frac{1}{2}f''(3)=6$	
		$f'(3)=2$		1
3	$f(3)=7$		$f(3, 3, 5)=8$	
		$f(3, 5)=18$		1
5	43		$f(3, 5, 13)=18$	
		198		1
13	1627		42	
		1122		1
27	17335		71	
		2613		1
34	35626		101	
		4229		
43	73687			

Here the third differences are constant, equal to 1.

Now
$$\begin{aligned} f(3,\ 5,\ 13) &= 42-24\times 1 = 18 \\ f(3,\ 5) &= 198-10\times 18 = 18 \\ f(3) &= 43-\ 2\times 18 = 7 \\ f(3,\ 3,\ 5) &= 18-10\times 1 = 8 \\ f'(3) &= 18-\ 2\times 8 = 2 \\ \tfrac{1}{2}f''(3) &= 8-\ 2\times 1 = 6. \end{aligned}$$

Hence substituting these values in the above table and applying Newton's formula we get

$$f(x)=7+2(x-3)+6(x-3)^2+(x-3)^3.$$

EXAMPLES

1. Shew that the divided differences of $f(x)$ are unaltered if the arguments be each increased by the same constant C, while the corresponding tabular values are left unchanged.

[*C.U. M.A. & M.Sc., 1939*]

2. If the arguments be each multiplied by the same constant C, while the tabular values remain unchanged, shew that the divided difference of the nth order is multiplied by C^{-n}.

3. If $f(x)=x^{n+1}$, shew that

$$f(x_0x_1x_2\ldots\ldots x_n)=x_0+x_1+x_2+\ldots\ldots+x_n.$$

[*C. U. M.A. & M.Sc., 1934*]

4. Find the value of $f(19)$ from the following table:

x	11	17	21	23	31
$f(x)$	14646	83526	194486	279846	923526.

5. Construct a table of divided differences from the following data:

x	0	5	8	2	25
$f(x)$	-179	1	13	87	7221.

Extend the table to include arguments $x=5$, repeated as many times as may be necessary, and thus find $f(x)$ in powers of $(x-5)$.

6. Find an expression for $f(x)$ from the following data:

x	0	1	3	5	6	9
$f(x)$	-18	0	0	-248	0	13104.

7. Find an approximate value of $\log_{10} 4{\cdot}01$ from the following data:

x	4·0002	4·0104	4·0233	4·0294
$f(x)$	0·6020817	·6031877	·6045824	·6052404.

8. If n_1 arguments are equal to x_1, n_2 arguments equal to x_2, ..., n_p arguments equal to x_p, shew that

$$f(x_1x_1\ldots\ldots x_px_p)=\int_0^1 dt_1\int_{0.}^{t_1}dt_2\ldots\ldots\int_0^{t_{p-2}}\phi(t)f^{(n-1)}(y)\,dt_{p-1},$$

where $y=(1-t_1)x_1+(t_1-t_2)x_2+\ldots+(t_{p-2}-t_{p-1})x_{p-1}+t_{p-1}x_p$, and

$$\phi(t)=\frac{(1-t_1)^{n_1-1}(t_1-t_2)^{n_2-1}\ldots\ldots(t_{p-2}-t_{p-1})^{n_{p-1}-1}t_{p-1}{}^{n_p-1}}{(n_1-1)!\,(n_2-1)!\ldots\ldots(n_p-1)!}.$$

9. Form a table of divided differences from the following values:

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

and extend it to include the values of $f(x)$ for $x=3$ and $x=14$.

10. Express the confluent divided difference $f(aaabbc)$ as the quotient of two six-row determinants.

11. Prove that

$$f(x_1x_2\ldots\ldots x_n)=\int\ldots\ldots\int f^{(n-1)}(t_1x_1+t_2x_2+\ldots+t_nx_n)\,dt_1dt_2\ldots dt_n,$$

where the integration is extended to all positive values, including zero, which satisfy $t_1+t_2+\ldots\ldots+t_n=1$.

[Genocchi]

CHAPTER VI

CENTRAL DIFFERENCES

6.1. Central Differences. Differences have hitherto been used which lie on a line sloping upwards or downwards from $f(a)$ in the difference table. Now we shall employ differences which lie on a horizontal line in the table. Instead of keeping these as they are, we shall use the modified notation as employed by W. F. Sheppard.* He introduces a symbol δ which is equivalent to $\Delta E^{-\frac{1}{2}}$. The operator δ is known as the *Central Difference Operator*.

Since $$\delta \equiv \Delta E^{-\frac{1}{2}} \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

we have $$\Delta \equiv \delta E^{\frac{1}{2}}.$$

Thus $$\Delta u_0 = \delta u_{\frac{1}{2}},\ \Delta^2 u_0 = \delta^2 u_1,\ \Delta u_1, = \delta u_{\frac{3}{2}},\ \text{etc.}$$

With this modified notation, the difference table can be put thus:

$a-3\omega$	u_{-3}						
		$\delta u_{-\frac{5}{2}}$					
$a-2\omega$	u_{-2}		$\delta^2 u_{-2}$				
		$\delta u_{-\frac{3}{2}}$		$\delta^3 u_{-\frac{3}{2}}$			
$a-\omega$	u_{-1}		$\delta^2 u_{-1}$		$\delta^4 u_{-1}$		
		$\delta u_{-\frac{1}{2}}$		$\delta^3 u_{-\frac{1}{2}}$		$\delta^5 u_{-\frac{1}{2}}$	
a	u_0		$\delta^2 u_0$		$\delta^4 u_0$		$\delta^6 u_0$
		$\delta u_{\frac{1}{2}}$		$\delta^3 u_{\frac{1}{2}}$		$\delta^5 u_{\frac{1}{2}}$	
$a+\omega$	u_1		$\delta^2 u_1$		$\delta^4 u_1$		
		$\delta u_{\frac{3}{2}}$		$\delta^3 u_{\frac{3}{2}}$			
$a+2\omega$	u_2		$\delta^2 u_2$				
		$\delta u_{\frac{5}{2}}$					
$a+3\omega$	u_3						

It may be noted that the horizontal line through u_x contains the differences of u_x of even order, and the horizontal line between u_x and u_{x+1} contains differences of $u_{x+\frac{1}{2}}$ of odd order.

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* *Proc. London. Math. Soc.* 31 (1899), 459

The arithmetic mean of two successive entries or differences in the same column is represented by the symbol μ, and is defined as

$$\mu \equiv \tfrac{1}{2}(E^{\frac{1}{2}} + E^{-\frac{1}{2}}).$$

Thus

$$\mu u_0 = \tfrac{1}{2}(u_{-\frac{1}{2}} + u_{\frac{1}{2}}), \qquad \mu u_{\frac{1}{2}} = \tfrac{1}{2}(u_0 + u_1).$$

$$\mu u_r = \tfrac{1}{2}(u_{r-\frac{1}{2}} + u_{r+\frac{1}{2}}), \qquad \mu u_{r+\frac{1}{2}} = \tfrac{1}{2}(u_r + u_{r+1}).$$

$$\mu\delta u_0 = \tfrac{1}{2}(\delta u_{-\frac{1}{2}} + \delta u_{\frac{1}{2}}), \qquad \mu\delta^2 u_{\frac{1}{2}} = \tfrac{1}{2}(\delta^2 u_0 + \delta^2 u_1).$$

$$\mu\delta u_r = \tfrac{1}{2}(\delta u_{r-\frac{1}{2}} + \delta u_{r+\frac{1}{2}}), \qquad \mu\delta u_{r+\frac{1}{2}} = \tfrac{1}{2}(\delta u_r + \delta u_{r+1}).$$

Also

$$\mu\delta = \tfrac{1}{2}(E - E^{-1}) = \frac{\Delta}{1+\Delta}\left(1 + \frac{\Delta}{2}\right),$$

and

$$\mu\delta^{2r+1} = \tfrac{1}{2}\{E^{-(r-1)} - E^{-(r+1)}\}\Delta^{2r} = \frac{\Delta^{2r+1}}{E^{r+1}}\left(1 + \frac{\Delta}{2}\right).$$

These mean differences can be inserted to fill in the gaps between the differences from which they are derived. The table will then stand thus:

a	u_0	$\mu\delta u_0$	$\delta^2 u_0$	$\mu\delta^3 u_0$	$\delta^4 u_0$	$\mu\delta^5 u_0$	$\delta^6 u_0$
	$\mu u_{\frac{1}{2}}$	$\delta u_{\frac{1}{2}}$	$\mu\delta^2 u_{\frac{1}{2}}$	$\delta^3 u_{\frac{1}{2}}$	$\mu\delta^4 u_{\frac{1}{2}}$	$\delta^5 u_{\frac{1}{2}}$	$\mu\delta^6 u_{\frac{1}{2}}$
$a+\omega$	u_1	$\mu\delta u_1$	$\delta^2 u_1$	$\mu\delta^3 u_1$	$\delta^4 u_1$	$\mu\delta^5 u_1$	$\delta^6 u_1$

……………………………………………………

Many formulæ can be put in very compact forms with the notation of central differences. Here we are not restricted to leading differences only. We can use the central differences in a horizontal line or in zigzag lines.

6.2. Newton-Gauss Formula of Interpolation. We have from Newton's formula §5.6

$$(1)\quad f(u) = f(x_1) + (u-x_1)f(x_1x_2) + (u-x_1)(u-x_2)f(x_1x_2x_3) + \ldots$$
$$+ (u-x_1)(u-x_2)\ldots(u-x_{n-1})f(x_1x_2\ldots x_n) + R_n(u),$$

where

$$R_n(u) = (u-x_1)(u-x_2)\ldots(u-x_n)f(ux_1x_2\ldots x_n).$$

Put $x_1=a,\quad x_{2r}=a+r\omega,\; x_{2r+1}=a-r\omega.$

Let $u=a+x\omega$, then $f(u)=f(a+x\omega)=u_x$.

By §5.2 we have

$$f(x_1x_2\ldots x_{2r+1})=\frac{1}{(2r)!\,\omega^{2r}}\,\Delta^{2r}f(a-r\omega)=\frac{1}{(2r)!\,\omega^{2r}}\,\delta^{2r}u_0$$

and

$$f(x_1x_2\ldots x_{2r+2})=\frac{1}{(2r+1)!\,\omega^{2r+1}}\Delta^{2r+1}f(a-r\omega)$$

$$=\frac{1}{(2r+1)!\,\omega^{2r+1}}\delta^{2r+1}u_{\frac{1}{2}}.$$

Also

$$(u-x_1)(u-x_2)\ldots(u-x_{2r})$$

$$=(u-x_{2r-1})(u-x_{2r-3})\ldots(u-x_3)(u-x_1)(u-x_2)(u-x_4)\ldots(u-x_{2r})$$

$$=(x+r-1)(x+r-2)\ldots(x+1)x(x-1)(x-2)\ldots(x-r)\omega^{2r}$$

$$=\binom{x+r-1}{2r}(2r)!\,\omega^{2r}.\,^{*}$$

Hence $(u-x_1)(u-x_2)\ldots(u-x_{2r})f(x_1x_2\ldots x_{2r+1})$

$$=\binom{x+r-1}{2r}\Delta^{2r}f(a-r\omega)=\binom{x+r-1}{2r}\delta^{2r}u_0.$$

Similarly $(u-x_1)(u-x_2)\ldots(u-x_{2r+1})f(x_1x_2\ldots x_{2r+2})$

$$=\binom{x+r}{2r+1}\Delta^{2r+1}f(a-r\omega)=\binom{x+r}{2r+1}\delta^{2r+1}u_{\frac{1}{2}}.$$

Substituting these values in (1) we have

$$(2)\quad f(a+x\omega)=f(a)+x\Delta f(a)+\frac{x(x-1)}{2!}\Delta^2 f(a-\omega)$$

$$+\frac{(x+1)x(x-1)}{3!}\Delta^3 f(a-\omega)+\frac{(x+1)x(x-1)(x-2)}{4!}\Delta^4 fa-2\omega)$$

$$+\frac{(x+2)(x+1)x(x-1)(x-2)}{5!}\Delta^5 f(a-2\omega)+\ldots+R_n(u)\,;$$

---

* The symbol $\binom{n}{r}$ stands for the binomial coefficient

$$\frac{n(n-1)\,(n-2)\ldots\ldots(n-r+1)}{r!}.$$

or, in central difference notation

$$(3)\quad f(a+x\omega)=u_x=u_0+x\delta u_{\frac{1}{2}}+\binom{x}{2}\delta^2u_0+\binom{x+1}{3}\delta^3u_{\frac{1}{2}}$$

$$+\binom{x+1}{4}\delta^4u_0+\binom{x+2}{5}\delta^5u_{\frac{1}{2}}+\ldots\ldots$$

$$+\binom{x+r-1}{2r}\delta^{2r}u_0+\binom{x+r}{2r+1}\delta^{2r+1}u_{\frac{1}{2}}+\ldots+R_n(x).$$

This is Newton-Gauss Forward Formula. In it the following zigzag scheme of differences have been employed :

$$\begin{tikzcd}
a & u_0 \arrow[dr, dash, dotted] & & \delta^2u_0 \arrow[dr, dash, dotted] & & \delta^4u_0 \arrow[dr, dash, dotted] & & \\
 & & \delta u_{\frac{1}{2}} \arrow[ur, dash, dotted] & & \delta^3u_{\frac{1}{2}} \arrow[ur, dash, dotted] & & \delta^5u_{\frac{1}{2}} \arrow[ur, dash, dotted] & \\
a+\omega & u_1 & & & & & &
\end{tikzcd}$$

If $n=2m$, we have by §5.7(3)

$$R_{2m}(u)=\binom{x+m-1}{2m}\omega^{2m}f^{(2m)}(\xi).$$

so that

$$(4)\quad f(a+x\omega)=u_x=u_0+\sum_{r=0}^{m-1}\binom{x+r}{2r+1}\delta^{2r+1}u_{\frac{1}{2}}$$

$$+\sum_{r=1}^{m-1}\binom{x+r-1}{2r}\delta^{2r}u_0+\binom{x+m-1}{2m}\omega^{2m}f^{(2m)}(\xi),$$

where ξ, u lie in the interval $\{a-(m-1)\omega,\ a+m\omega\}$.

If $n=2m+1$, we have

$$(5)\quad f(a+x\omega)=u_x=u_0+\sum_{r=0}^{m-1}\binom{x+r}{2r+1}\delta^{2r+1}u_{\frac{1}{2}}$$

$$+\sum_{r=1}^{m}\binom{x+r-1}{2r}\delta^{2r}u_0+\binom{x+m}{2m+1}\omega^{2m+1}f^{(2m+1)}(\xi),$$

where ξ, u lie in the interval $(a-m\omega,\ a+m\omega)$.

Example. Given the values of log sin θ for $\theta=6°, 7°, \ldots . 12°$, compute the value of log sin 9° 22′.

Forming the table of differences, we have

6°	9·01923					
		6666				
7°	9·08589		−899			
		5767		209		
8°	9·14356		−690		−62	
		5077		147		17
9°	9·19433		−543		−45	
		4534		102		19
10°	9·23967		−441		−26	
		4093		76		
11°	9·28060		−365			
		3728				
12°	9·31788					

Here $a=9°$, $\omega=1°$ and $x=\frac{22}{60}=0{\cdot}36667$.

Hence by Newton-Gauss formula §6.2(2) we get

log sin 9°22′ =		9·19433	=	9·19433
	+0·36667 ×	4534	+	1662.5
	−0·11611 ×	−543	+	63
	−0·05289 ×	102	−	5.4
	+0·02160 ×	−45	−	1.0
	+0·01022 ×	19	+	.2
			=	9·21152.3

$\therefore$ log sin 9°22′ = 9·21152.3.

6.3. Newton-Gauss Backward Formula. Changing the sign of ω in the formulæ of §6.2, backward formulæ can be obtained where x is measured in a negative direction from $f(a)$. These can be directly obtained thus:

Put $x_1=a$, $x_{2r}=a-r\omega$, $x_{2r+1}=a+r\omega$ and $u=a-x\omega$. Then

$$f(x_1 x_2 \ldots x_{2r+1})=\frac{1}{(2r)\,!\,\omega^{2r}}\Delta^{2r}f(a-r\omega)=\frac{1}{(2r)\,!\,\omega^{2r}}\delta^{2r}u_0$$

$$f(x_1 x_2 \ldots x_{2r+2})=\frac{1}{(2r+1)\,!\,\omega^{2r+1}}\Delta^{2r+1}f(a-\overline{r+1}\omega)$$

$$=\frac{1}{(2r+1)\,!\,\omega^{2r+1}}\delta^{2r+1}u_{-\frac{1}{2}}.$$

Also $(u-x_1)(u-x_2)\ldots\ldots(u-x_{2r})$

$$=(-1)^{2r}(x+r-1)(x+r-2)\ldots x\ldots(x-r)\omega^{2r}$$

$$=\binom{x+r-1}{2r}(2r)!\,\omega^{2r},$$

and $(u-x_1)(u-x_2)\ldots\ldots(u-x_{2r+1})$

$$=(-1)^{2r+1}(x+r)(x+r-1)\ldots(x-r)\omega^{2r+1}$$

$$=-\binom{x+r}{2r+1}(2r+1)!\,\omega^{2r+1}.$$

Hence

$$(u-x_1)(u-x_2)\ldots(u-x_{2r})f(x_1x_2\ldots x_{2r+1})$$

$$=\binom{x+r-1}{2r}\Delta^{2r}f(a-r\omega)=\binom{x+r-1}{2r}\delta^{2r}u_0.$$

and $(u-x_1)(u-x_2)\ldots(u-x_{2r+1})f(x_1x_2\ldots x_{2r+2})$

$$=-\binom{x+r}{2r+1}\Delta^{2r+1}f(a-\overline{r+1}\,\omega)=-\binom{x+r}{2r+1}\delta^{2r+1}u_{-\frac{1}{2}}.$$

Substituting these values we have

$$(1)\ f(a-x\omega)=f(a)-x\Delta f(a-\omega)+\binom{x}{2}\Delta^2 f(a-\omega)-\binom{x+1}{3}\Delta^3 f(a-2\omega)$$

$$+\binom{x+1}{4}\Delta^4 f(a-2\omega)-\binom{x+2}{5}\Delta^5 f(a-3\omega)+\ldots$$

$$+\binom{x+r-1}{2r}\Delta^{2r}f(a-r\omega)-\binom{x+r}{2r+1}\Delta^{2r+1}f(a-\overline{r+1}\,\omega)+\ldots+R_n(u).$$

This is Newton-Gauss Backward Formula.

Expressed in terms of Central differences, the formula becomes

$$(2)\ f(a-x\omega)=u_{-x}=u_0-x\delta u_{-\frac{1}{2}}+\binom{x}{2}\delta^2 u_0-\binom{x+1}{3}\delta^3 u_{-\frac{1}{2}}$$

$$+\binom{x+1}{4}\delta^4 u_0-\binom{x+2}{5}\delta^5 u_{-\frac{1}{2}}+\ldots$$

$$+\binom{x+r-1}{2r}\delta^{2r}u_0-\binom{x+r}{2r+1}\delta^{2r+1}u_{-\frac{1}{2}}+\ldots+R_n(u).$$

If $n=2m$, we have

$$R_n(u)=\binom{x+m-1}{2m}\omega^{2m}f^{(2m)}(\xi)$$

where ξ, u lie in the interval $(a-m\omega,\ a+\overline{m-1}\omega)$.

If $n=2m+1$, $R_n(u)=-\binom{x+m}{2m+1}\omega^{2m+1}f^{(2m+1)}(\xi)$,

where ξ, u lie in the interval $(a-m\omega,\ a+m\omega)$.

Again putting $u=a+x\omega$ with $x_1=a$, $x_{2r}=a-r\omega$ and $x_{2r+1}=a+r\omega$ we get

$$f(a+x\omega)=u_x=u_0+x\delta u_{-\frac{1}{2}}+\binom{x+1}{2}\delta^2 u_0+\binom{x+1}{3}\delta^3 u_{-\frac{1}{2}}$$

$$+\binom{x+2}{4}\delta^4 u_0+\ldots+\binom{x+r}{2r}\delta^{2r}u_0+\binom{x+r}{2r+1}\delta^{2r+1}u_{-\frac{1}{2}}+\ldots+R_n(u).$$

Thus for $n=2m$

$$(3)\quad f(a+x\omega)=u_x=u_0+\sum_{r=0}^{m-1}\binom{x+r}{2r+1}\delta^{2r+1}u_{-\frac{1}{2}}+\sum_{r=1}^{m-1}\binom{x+r}{2r}\delta^{2r}u_0$$

$$+\binom{x+m}{2m}\omega^{2m}f^{(2m)}(\xi);$$

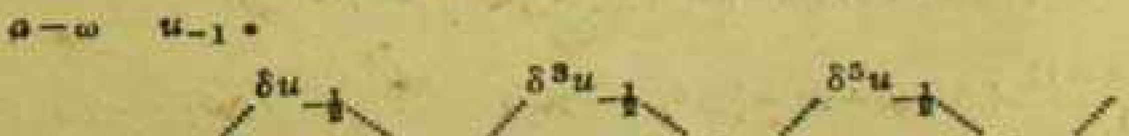
while for $n=2m+1$

$$(4)\quad f(a+x\omega)=u_x=u_0+\sum_{r=0}^{m-1}\binom{x+r}{2r+1}\delta^{2r+1}u_{-\frac{1}{2}}+\sum_{r=1}^{m}\binom{x+r}{2r}\delta^{2r}u_0$$

$$+\binom{x+m}{2m+1}\omega^{2m+1}f^{(2m+1)}(\xi).$$

It should be noted that Gauss' forward formula §6.2(5) has the same remainder term as the backward formula (4) if the last difference in each be of the same even order.

Here the following zigzag scheme has been used:


$a-\omega$
u_{-1}
$\delta u_{-\frac{1}{2}}$
$\delta^3 u_{-\frac{1}{2}}$
$\delta^5 u_{-\frac{1}{2}}$
a
u_0
$\delta^2 u_0$
$\delta^4 u_0$
$\delta^6 u_0$

6.4. Steffensen's Interpolation Formula. We have from Newton-Gauss forward formula §6.2 (5)

$$(1)\quad u_x = u_0 + \sum_{r=0}^{m-1}\binom{x+r}{2r+1}\delta^{2r+1}u_{\frac{1}{2}} + \sum_{r=1}^{m}\binom{x+r-1}{2r}\delta^{2r}u_0$$

$$+\binom{x+m}{2m+1}\omega^{2m+1}f^{(2m+1)}(\xi),$$

where ξ, u lie in the interval $(a-m\omega, a+m\omega)$.

Now

$$\sum_{r=0}^{m-1}\binom{x+r}{2r+1}\delta^{2r+1}u_{\frac{1}{2}} + \sum_{r=1}^{m}\binom{x+r-1}{2r}\delta^{2r}u_0$$

$$=\sum_{r=1}^{m}\left\{\binom{x+r-1}{2r-1}\delta^{2r-1}u_{\frac{1}{2}} + \binom{x+r-1}{2r}\left(\delta^{2r-1}u_{\frac{1}{2}} - \delta^{2r-1}u_{-\frac{1}{2}}\right)\right\}$$

$$=\sum_{r=1}^{m}\binom{x+r-1}{2r-1}\left\{\frac{x+r}{2r}\delta^{2r-1}u_{\frac{1}{2}} - \frac{x-r}{2r}\delta^{2r-1}u_{-\frac{1}{2}}\right\}$$

$$=\sum_{r=1}^{m}\left\{\binom{x+r}{2r}\delta^{2r-1}u_{\frac{1}{2}} - \binom{-x+r}{2r}\delta^{2r-1}u_{-\frac{1}{2}}\right\}.$$

Substituting these in (1) above we get Steffensen's formula

$$(2)\quad u_x = u_0 - \sum_{r=1}^{m}\binom{-x+r}{2r}\delta^{2r-1}u_{-\frac{1}{2}} + \sum_{r=1}^{m}\binom{x+r}{2r}\delta^{2r-1}u_{\frac{1}{2}}$$

$$+\binom{x+m}{2m+1}\omega^{2m+1}f^{(2m+1)}(\xi).$$

It employs the following scheme of differences :

$$\begin{array}{llllll}
a-\omega & u_{-1} & & & & \\
 & & \delta u_{-\frac{1}{2}} & \delta^3 u_{-\frac{1}{2}} & \delta^5 u_{-\frac{1}{2}} & \cdots \\
a & u_0 & & & & \\
 & & \delta u_{\frac{1}{2}} & \delta^3 u_{\frac{1}{2}} & \delta^5 u_{\frac{1}{2}} & \cdots \\
a+\omega & u_1 & & & &
\end{array}$$

6.5. Newton-Stirling Formula. Rearranging the terms of Gauss formula §6.2(2) in the form

$$f(a+x\omega) = f(a) + x\{\Delta f(a) - \tfrac{1}{2}\Delta^2 f(a-\omega)\} + \frac{x^2}{2!}\Delta^2 f(a-\omega)$$

$$+\frac{x(x^2-1)}{3!}\{\Delta^3 f(a-\omega) - \tfrac{1}{2}\Delta^4 f(a-2\omega)\} + \frac{x^2(x^2-1)}{4!}\Delta^4 f(a-2\omega) + \ldots$$

let us replace the differences of even order within the brackets by means of the following relations

$$\Delta^2 f(a-\omega) = \Delta f(a) - \Delta f(a-\omega),$$

$$\Delta^4 f(a-2\omega) = \Delta^3 f(a-\omega) - \Delta^3 f(a-2\omega),$$

$$\Delta^6 f(a-3\omega) = \Delta^5 f(a-2\omega) - \Delta^5 f(a-3\omega),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Delta^{2r} f(a-r\omega) = \Delta^{2r-1} f(a-\overline{r-1}\,\omega) - \Delta^{2r-1} f(a-r\omega).$$

Then we have

$$(1)\quad f(a+x\omega) = f(a) + x\frac{\Delta f(a)+\Delta f(a-\omega)}{2} + \frac{x^2}{2!}\Delta^2 f(a-\omega)$$

$$+\frac{x(x^2-1^2)}{3!}\,\frac{\Delta^3 f(a-\omega)+\Delta^3 f(a-2\omega)}{2} + \frac{x^2(x^2-1^2)}{4!}\Delta^4 f(a-2\omega)$$

$$+\frac{x(x^2-1^2)(x^2-2^2)}{5!}\,\frac{\Delta^5 f(a-2\omega)+\Delta^5 f(a-3\omega)}{2}$$

$$+\frac{x^2(x^2-1^2)(x^2-2^2)}{6!}\Delta^6 f(a-3\omega) + \dots$$

This formula was given by Newton* and afterwards studied by Stirling.†

In terms of Central differences it is put thus:

$$(2)\quad u_x = u_0 + x\mu\delta u_0 + \frac{x^2}{2!}\delta^2 u_0 + \frac{x(x^2-1^2)}{3!}\mu\delta^3 u_0 + \frac{x^2(x^2-1^2)}{4!}\delta^4 u_0$$

$$+\frac{x(x^2-1^2)(x^2-2^2)}{5!}\mu\delta^5 u_0 + \dots$$

This result is easily obtained by taking the Arithmetic mean of Gauss' forward formula §6.2(5) and backward formula §6.3(4).

---

* **Newton**—*Methodus Differentialis* (1711).

† **Stirling**—*Methodus Differentialis* (1730).

Thus

$$f(a+x\omega)=u_0+\tfrac{1}{2}\sum_{r=0}^{m-1}\binom{x+r}{2r+1}\left\{\delta^{2r+1}u_{\frac{1}{2}}+\delta^{2r+1}u_{-\frac{1}{2}}\right\}$$

$$+\tfrac{1}{2}\sum_{r=1}^{m}\left\{\binom{x+r-1}{2r}+\binom{x+r}{2r}\right\}\delta^{2r}u_0$$

$$+\binom{x+m}{2m+1}\omega^{2m+1}f^{(2m+1)}(\xi),$$

where ξ, u lie in the interval $(a-m\omega,\ a+m\omega)$.

Now $$\binom{x+r}{2r+1}=\frac{(x+r)(x+r-1)\ldots(x+1)x(x-1)\ldots(x-r)}{(2r+1)!}$$

$$=\frac{x(x^2-1^2)(x^2-2^2)\ldots(x^2-r^2)}{(2r+1)!},$$

and $$\binom{x+r-1}{2r}+\binom{x+r}{2r}=\frac{2x(x+r-1)\ldots x\ldots(x-r+1)}{(2r)!}$$

$$=\frac{2x^2(x^2-1^2)(x^2-2^2)\ldots(x^2-\overline{r-1}^2)}{(2r)!}.$$

Hence

$$f(a+x\omega)=u_x=u_0+\sum_{r=0}^{m-1}\frac{x(x^2-1^2)(x^2-2^2)\ldots(x^2-r^2)}{(2r+1)!}\mu\delta^{2r+1}u_0$$

(8) $$+\sum_{r=1}^{m}\frac{x^2(x^2-1^2)(x^2-2^2)\ldots(x^2-\overline{r-1}^2)}{(2r)!}\delta^{2r}u_0$$

$$+\frac{x(x^2-1^2)(x^2-2^2)\ldots(x^2-m^2)}{(2m+1)!}\omega^{2m+1}f^{(2m+1)}(\xi).$$

The differences employed here lie on a horizontal line through u_0, namely,

$$a,\quad u_0,\quad \mu\delta u_0,\quad \delta^2u_0,\quad \mu\delta^3u_0,\quad \delta^4u_0,\quad \mu\delta^5u_0,\ldots$$

Stirling's formula gives more accurate results when x lies in the range $-\frac{1}{2}$ to $+\frac{1}{2}$, *i.e.*, $\frac{1}{2}$ on each side of the line through u_0.

Example. Consider the example of p. 147.

Here the differences employed lie on the horizontal line through 9°, which is as follows :—

9° 9·19433, 4805.5, −543, 124.5, −45, 18.

Hence by Newton-Stirling formula §6.5(2) we get

$$
\begin{array}{llrcr}
\log\sin 9^\circ 22' = & & 9{\cdot}19433 & = & 9{\cdot}19433 \\
 & +0{\cdot}36667\times & 4805.5 & + & 1762.0 \\
 & +0{\cdot}06722\times & -\ 543 & - & 36.5 \\
 & -0{\cdot}05289\times & 124.5 & - & 6.6 \\
 & -0{\cdot}00485\times & -\ \ 45 & + & 0.2 \\
 & +0{\cdot}01022\times & 18 & + & 0.2 \\
\cline{4-5}
 & & & = & 9{\cdot}21152.8
\end{array}
$$

$$\therefore\ \log\sin 9^\circ 22' = 9{\cdot}21152.8,$$

which agrees with the value obtained in p. 147.

6.6. Newton-Bessel Formula of Interpolation. By Gauss' formula §6.2 (4) which end with a difference of odd order, we have

$$(1)\quad f(a+x\omega)=u_0+\sum_{r=0}^{m-1}\binom{x+r}{2r+1}\delta^{2r+1}u_{\frac{1}{2}}+\sum_{r=1}^{m-1}\binom{x+r-1}{2r}\delta^{2r}u_0$$

$$+\binom{x+m-1}{2m}\omega^{2m}f^{(2m)}(\xi).$$

The corresponding backward formula § 6.3(3) with the initial argument $a+\omega$ is obtained by writing $x-1$ in place of x. Thus

$$(2)\quad f(a+x\omega)=u_1+\sum_{r=0}^{m-1}\binom{x+r-1}{2r+1}\delta^{2r+1}u_{\frac{1}{2}}$$

$$+\sum_{r=1}^{m-1}\binom{x+r-1}{2r}\delta^{2r}u_1+\binom{x+m-1}{2m}\omega^{2m}f^{(2m)}(\xi).$$

Now taking the Arithmetic mean of (1) and (2), we get

$$(3)\quad f(a+x\omega)=\mu u_{\frac{1}{2}}+\sum_{r=0}^{m-1}\frac{x-\frac{1}{2}}{2r+1}\binom{x+r-1}{2r}\delta^{2r+1}u_{\frac{1}{2}}$$

$$+\sum_{r=1}^{m-1}\binom{x+r-1}{2r}\mu\delta^{2r}u_{\frac{1}{2}}+\binom{x+m-1}{2m}\omega^{2m}f^{(2m)}(\xi),$$

where ξ, u lie in the interval $(a-\overline{m-1}\omega,\ a+m\omega)$.

This formula was first given by Newton and afterwards used by Bessel and is known as Newton-Bessel Formula. Here x should lie in the range 0 to 1.

Again since

$$\mu u_{\frac{1}{2}}+(x-\tfrac{1}{2})\delta u_{\frac{1}{2}}=u_0+x\delta u_{\frac{1}{2}},$$

the formula (3) can be put as

$$(4)\qquad u_x=u_0+x\delta u_{\frac{1}{2}}+\sum_{r=1}^{m-1}\frac{x-\frac{1}{2}}{2r+1}\binom{x+r-1}{2r}\delta^{2r+1}u_{\frac{1}{2}}$$

$$+\sum_{r=1}^{m-1}\binom{x+r-1}{2r}\mu\delta^{2r}u_{\frac{1}{2}}+\binom{x+m-1}{2m}\omega^{2m}f^{(2m)}(\xi).$$

A more symmetric and convenient form is obtained by putting $x=y+\frac{1}{2}$. Thus

$$(5)\qquad u_{y+\frac{1}{2}}=\mu u_{\frac{1}{2}}+y\delta u_{\frac{1}{2}}+\sum_{r=1}^{m-1}\frac{y}{2r+1}\binom{y+\frac{1}{2}(2r-1)}{2r}\delta^{2r+1}u_{\frac{1}{2}}$$

$$+\sum_{r=1}^{m-1}\binom{y+\frac{1}{2}(2r-1)}{2r}\mu\delta^{2r}u_{\frac{1}{2}}+\binom{y+\frac{1}{2}(2m-1)}{2m}\omega^{2m}f^{(2m)}(\xi).$$

Writing the first few terms in full we have

$$(6)\qquad u_{y+\frac{1}{2}}=\mu u_{\frac{1}{2}}+y\delta u_{\frac{1}{2}}+\frac{y^2-\frac{1}{4}}{2!}\mu\delta^2u_{\frac{1}{2}}+\frac{y(y^2-\frac{1}{4})}{3!}\delta^3u_{\frac{1}{2}}$$

$$+\frac{(y^2-\frac{1}{4})(y^2-\frac{9}{4})}{4!}\mu\delta^4u_{\frac{1}{2}}+\frac{y(y^2-\frac{1}{4})(y^2-\frac{9}{4})}{5!}\delta^5u_{\frac{1}{2}}+\dots$$

The differences employed here are taken from the scheme

$$\begin{array}{llccccc}
a & u_0 & & & & & \\
 & & \delta u_{\frac{1}{2}} & \mu\delta^2u_{\frac{1}{2}} & \delta^3u_{\frac{1}{2}} & \mu\delta^4u_{\frac{1}{2}} & \delta^5u_{\frac{1}{2}},\dots \\
a+\omega & u_1 & & & & &
\end{array}$$

Example. Take the example of p. 147.

Here the following scheme of differences is employed:

$$\begin{array}{llccccc}
9^\circ & 9{\cdot}19433 & & & & & \\
 & & 4534, & -492, & 102, & -35.5, & 19. \\
10^\circ & & & & & &
\end{array}$$

Hence by Newton-Bessel formula §6.6 (4) we have

log sin 9°22′ =		9·19433	=	9·19433
	+0·36667 ×	4534	+	1662.5
	−0·11611 ×	− 492	+	57.1
	+0·00516 ×	102	+	0.5
	+0·02160 ×	− 35.5	−	0.8
	−0·00057 ×	19	−	0.0
			=	9·21152.3

∴ log sin 9°22′ = 9·21152.3.

6.7. Everett's Formula of Interpolation. In this formula only even differences on horizontal lines through u_0 and u_1 are used. It is obtained by eliminating odd differences from Newton-Gauss formula.

We have from §6.2(4)

$$u_x = u_0 + x\delta u_{\frac{1}{2}} + \sum_{r=1}^{m-1}\left\{\binom{x+r-1}{2r}\delta^{2r}u_0 + \binom{x+r}{2r+1}\delta^{2r+1}u_{\frac{1}{2}}\right\}$$

$$+\binom{x+m-1}{2m}\omega^{2m}f^{(2m)}(\xi).$$

Now

$$\binom{x+r-1}{2r}\delta^{2r}u_0 + \binom{x+r}{2r+1}\delta^{2r+1}u_{\frac{1}{2}}$$

$$=\frac{(x+r-1)\ldots\ldots(x-r)}{(2r+1)\,!}\left\{(2r+1)\delta^{2r}u_0 + (x+r)\delta^{2r+1}u_{\frac{1}{2}}\right\}$$

$$=\frac{(x+r-1)\ldots\ldots(x-r)}{(2r+1)\,!}\left\{(2r+1)\delta^{2r}u_0 + (x+r)\,(\delta^{2r}u_1 - \delta^{2r}u_0)\right\}$$

$$=\binom{x+r}{2r+1}\delta^{2r}u_1 - \binom{x+r-1}{2r+1}\delta^{2r}u_0.$$

Hence

$$(1)\quad u_x = u_0 + x\delta u_{\frac{1}{2}} + \sum_{r=1}^{m-1}\left\{\binom{x+r}{2r+1}\delta^{2r}u_1 - \binom{x+r-1}{2r+1}\delta^{2r}u_0\right\}$$

$$+\binom{x+m-1}{2m}\omega^{2m}f^{(2m)}(\xi).$$

This is Everett's Formula.

Again since $\quad\binom{x+r-1}{2r+1} = -\binom{1-x+r}{2r+1}$

and $\quad u_0 + x\delta u_{\frac{1}{2}} = u_0(1-x) + xu_1,$

putting $1-x=y$, we have an elegant form of Everett's formula*

---

* **Everett**, *Brit. Assoc. Rep.* (1900), p. 648; *J. I. A.* 35, p. 452 (1901). See also **Laplace**, *Theorie anal. des Prob.*, p. 15.

(2) $$u_x = xu_1 + \sum_{r=1}^{m-1} \binom{x+r}{2r+1} \delta^{2r} u_1$$
$$+ yu_0 + \sum_{r=1}^{m-1} \binom{y+r}{2r+1} \delta^{2r} u_0$$
$$+ \binom{x+m-1}{2m} \omega^{2m} f^{(2m)}(\xi),$$

where ξ lies in the interval $(a - \overline{m-1}\omega,\ a + m\omega)$.

A second formula of Everett employing odd differences which lie on horizontal lines through $u_{-\frac{1}{2}}$ and $u_{\frac{1}{2}}$, is obtained thus. We have from Newton-Gauss backward formula §6.3(4)

$$u_x = u_0 + \sum_{r=1}^{m} \left\{ \binom{x+r-1}{2r-1} \delta^{2r-1} u_{-\frac{1}{2}} + \binom{x+r}{2r} \delta^{2r} u_0 \right\}$$
$$+ \binom{x+m}{2m+1} \omega^{2m+1} f^{(2m+1)}(\xi).$$

Now

$$\binom{x+r-1}{2r-1} \delta^{2r-1} u_{-\frac{1}{2}} + \binom{x+r}{2r} \delta^{2r} u_0$$
$$= \binom{x+r-1}{2r-1} \delta^{2r-1} u_{-\frac{1}{2}} + \binom{x+r}{2r} (\delta^{2r-1} u_{\frac{1}{2}} - \delta^{2r-1} u_{-\frac{1}{2}})$$
$$= \binom{x+r}{2r} \delta^{2r-1} u_{\frac{1}{2}} - \binom{x+r-1}{2r} \delta^{2r-1} u_{-\frac{1}{2}}$$
$$= \binom{x+r}{2r} \delta^{2r-1} u_{\frac{1}{2}} - \binom{r-x}{2r} \delta^{2r-1} u_{-\frac{1}{2}}.$$

Substituting this in the above formula we have

$$u_x = u_0 + \sum_{r=1}^{m} \left\{ \binom{x+r}{2r} \delta^{2r-1} u_{\frac{1}{2}} - \binom{r-x}{2r} \delta^{2r-1} u_{-\frac{1}{2}} \right\}$$
$$+ \binom{x+m}{2m+1} \omega^{2m+1} f^{(2m+1)}(\xi).$$

Hence

$$(8) \quad u_{p-\frac{1}{2}} = u_0 + \sum_{r=1}^{m} \binom{p+\frac{1}{2}(2r-1)}{2r} \delta^{2r-1} u_{\frac{1}{2}} - \sum_{r=1}^{m} \binom{q+\frac{1}{2}(2r-1)}{2r} \delta^{2r-1} u_{-\frac{1}{2}} + \binom{p+\frac{1}{2}(2m-1)}{2m+1} \omega^{2m+1} f^{(2m+1)}(\xi)$$

where $p = x + \frac{1}{2}$ and $q = \frac{1}{2} - x$.

This is Everett's second formula. Both formulæ, more particularly the first, are specially adapted for use in statistical work.

Example. Interpolate for the age of 37 from the following data:

Age	25,	30,	35,	40,	45,	50,	55
Entry	1431,	1662,	1961,	2351,	2865,	3550,	4471.

Forming the difference table we have

x	age	u_x						
−2	25	1431						
			231					
−1	30	1662		68				
			299		23			
0	35	1961		91		10		
			390		33		4	
1	40	2351		124		14		0
			514		47		4	
2	45	2865		171		18		
			685		65			
3	50	3550		236				
			921					
4	55	4471						

For age 37, we have $x = \cdot 4$ and $y = \cdot 6$.

The coeff. for the first line are $\cdot 4, -\cdot 056, \cdot 01075$,

,, for the second line are $\cdot 6, -\cdot 064, \cdot 01165$.

The differences employed lie on the horizontal lines through 0 and 1. Hence by §6.7(2) the required value for age 37

$$= \cdot 4 \times 2351 - \cdot 056 \times 124 + \cdot 01075 \times 14$$
$$+ \cdot 6 \times 1961 - \cdot 064 \times 91 + \cdot 01165 \times 10$$
$$= 940 \cdot 4 - 6 \cdot 94 + \cdot 15 + 1176 \cdot 6 - 5 \cdot 82 + \cdot 11$$
$$= 2104 \cdot 5.$$

6.8. The chief advantage of Everett's formula lies in the fact that, for a subdivision of intervals, the terms are such that they occur both in the x-expansion, and in reverse order in the next y-expansion. So we need only calculate these once. If u_x is expanded seperately for each value of x, the whole advantage of the formula is lost. We illustrate it by an example below. This formula is specially applicable when the values to be interpolated lie near the middle of the series of given values. Everett's formula will be wholly unsuitable for interpolating values at the beginning or at the end of the given series and it will then be necessary to apply the general advancing difference formula of §4.4, which again is unsuitable for interpolating values near the middle unless a sufficient number of terms is given (see Ex. p. 94).

Example. Calculate for all ages from 35 to 45 from the data given in the example on p. 157.

We have from Everett's formula

$$u_x = xu_1 + \binom{x+1}{3}\delta^2 u_1 + \binom{x+2}{5}\delta^4 u_1 + \dots$$

$$+ yu_0 + \binom{y+1}{3}\delta^2 u_0 + \binom{y+2}{5}\delta^4 u_0 + \dots$$

and for the second interval

$$u_{1+y} = yu_2 + \binom{y+1}{3}\delta^2 u_2 + \binom{y+2}{5}\delta^4 u_2 + \dots$$

$$+ xu_1 + \binom{x+1}{3}\delta^2 u_1 + \binom{x+2}{5}\delta^4 u_1 + \dots$$

The second line in u_{1+y} is the same as the first line in u_x.

The values of x are ·2, ·4, ·6 and ·8 and consequently the corresponding values of y are ·8, ·6, ·4 and ·2. The coefficients of the terms in the first line of u_x are

·2, −·032, ·00634
·4, −·056, ·01075
·6, −·064, ·01165
·8, −·048, ·00806.

The terms involving differences of u_0, u_1 and u_2 are put in the following tabular form :

Age	x	1st term	2nd term	3rd term	Sum of 1st line	Sum of 2nd line	Result
	·2	392·2	−2·91	·06	389·35		
	·4	784·4	−5·09	·10	779·41		
	·6	1176·6	−5·82	·11	1170·89		
	·8	1568·8	−4·37	·08	1564·51		
36	·2	470·2	−3·97	·09	466·32	1564·51	2030·83
37	·4	940·4	−6·94	·15	933·61	1170·89	2104·50
38	·6	1410·6	−7·93	·16	1402·83	779·41	2182·24
39	·8	1880·8	−5·95	·11	1874·96	389·35	2264·31
41	·2	573·0	−5·47	·11	567·64	1874·96	2442·60
42	·4	1146·0	−9·57	·19	1136·62	1402·83	2539·45
43	·6	1719·0	−10·94	·21	1708·27	933·61	2641·88
44	·8	2292·0	−8·21	·15	2283·94	466·32	2750·26

The column for the "sum of 2nd line" is obtained by writing in reverse order the values in "sum of 1st line" from the previous tabulation.

6.9. So far we have deduced the central difference formulæ from Newton's general interpolation formula with Divided differences and other formulæ derived therefrom. We shall now try to deduce the formulæ direct on the assumption that u_x can be expressed in the form of a series and shall seek to determine the coefficients. Thus to express u_x in terms of ordinary central differences lying on the horizontal line through u_0, assume

$$u_x = A_0 u_0 + A_1 \mu\delta u_0 + A_2 \delta^2 u_0 + A_3 \mu\delta^3 u_0 + A_4 \delta^4 u_0 + \dots\dots$$
$$+ A_{2r-1}\mu\delta^{2r-1}u_0 + A_{2r}\delta^{2r}u_0 + \dots$$

Then

$$u_x = E^x u_0 = \left\{ A_0 + A_1 \frac{\Delta}{E}\left(1+\frac{\Delta}{2}\right) + A_2\frac{\Delta^2}{E} + A_3\frac{\Delta^3}{E^2}\left(1+\frac{\Delta}{2}\right) + \dots\right.$$
$$\left.\dots + A_{2r-1}\frac{\Delta^{2r-1}}{E^r}\left(1+\frac{\Delta}{2}\right) + A_{2r}\frac{\Delta^{2r}}{E^r} + \dots \right\} u_0$$

by substituting the values of central differences (§6.1).

Hence

$$(1)\quad E^x = A_0 + A_1\frac{\Delta}{E} + (\tfrac{1}{2}A_1 + A_2)\frac{\Delta^2}{E} + A_3\frac{\Delta^3}{E^2} + \dots$$

$$\dots + (\tfrac{1}{2}A_{2r-1} + A_{2r})\frac{\Delta^{2r}}{E^r} + A_{2r+1}\frac{\Delta^{2r+1}}{E^{r+1}} + \dots\dots$$

Multiplying both sides by E^r and equating the coefficient of Δ^{2r} we have

$$\tfrac{1}{2}A_{2r-1} + A_{2r} = \frac{(x+r)(x+r-1)\dots\dots(x-r+1)}{(2r)!}.$$

Again multiplying both sides of (1) by E^{r+1} and equating the coefficient of Δ^{2r+1} we have

$$\tfrac{1}{2}A_{2r-1} + A_{2r} + A_{2r+1} = \frac{(x+r+1)(x+r)\dots\dots(x-r+1)}{(2r+1)!}.$$

Hence by subtraction

$$A_{2r+1} = \frac{(x+r)(x+r-1)\dots\dots(x-r+1)(x-r)}{(2r+1)!}$$

$$= \frac{x(x^2-1^2)(x^2-2^2)\dots\dots(x^2-r^2)}{(2r+1)!},$$

whence $$A_{2r-1} = \frac{x(x^2-1^2)(x^2-2^2)\dots\dots(x^2-\overline{r-1}^2)}{(2r-1)!},$$

and $$A_{2r} = \frac{x^2(x^2-1^2)(x^2-2^2)\dots\dots(x^2-\overline{r-1}^2)}{(2r)!}.$$

Thus we get Newton-Stirling formula

$$u_x = u_0 + x\mu\delta u_0 + \frac{x^2}{2!}\delta^2 u_0 + \frac{x(x^2-1^2)}{3!}\mu\delta^3 u_0 + \dots\dots$$

6.10. Again to obtain Newton-Bessel formula which employs the Central Interval differences lying on the horizontal line through $\mu u_{\frac{1}{2}}$, assume

$$u_x = A_0\mu u_{\frac{1}{2}} + A_1\delta u_{\frac{1}{2}} + A_2\mu\delta^2 u_{\frac{1}{2}} + A_3\delta^3 u_{\frac{1}{2}} + \dots\dots$$

$$+ A_{2r-2}\mu\delta^{2r-2}u_{\frac{1}{2}} + A_{2r-1}\delta^{2r-1}u_{\frac{1}{2}} + A_{2r}\mu\delta^{2r}u_{\frac{1}{2}} + \dots$$

Then

$$(1)\quad E^x = A_0 + (\tfrac{1}{2}A_0 + A_1)\Delta + A_2\frac{\Delta^2}{E} + (\tfrac{1}{2}A_2 + A_3)\frac{\Delta^3}{E} + \dots\dots$$

$$+ (\tfrac{1}{2}A_{2r-2} + A_{2r-1})\frac{\Delta^{2r-1}}{E^{r-1}} + A_{2r}\frac{\Delta^{2r}}{E^r} + \dots\dots$$

Multiplying by E^{r-1} and equating the coefficient of Δ^{2r-1}, we have

$$\tfrac{1}{2}A_{2r-2} + A_{2r-1} = \frac{(x+r-1)(x+r-2)\dots\dots(x-r+1)}{(2r-1)!}.$$

Again multiplying (1) by E^r and equating the coefficient of Δ^{2r} we have

$$\tfrac{1}{2}A_{2r-2} + A_{2r-1} + A_{2r} = \frac{(x+r)(x+r-1)\dots\dots(x-r+1)}{(2r)!}.$$

Hence
$$A_{2r} = \frac{(x+r-1)(x+r-2)\dots\dots(x-r)}{(2r)!},$$

and
$$A_{2r-1} = \frac{2x-1}{2}.\frac{(x+r-2)\dots\dots(x-r+1)}{(2r-1)!}.$$

Hence

$$u_x = \mu u_{\frac{1}{2}} + (x-\tfrac{1}{2})\delta u_{\frac{1}{2}} + \frac{x(x-1)}{2!}\mu\delta^2 u_{\frac{1}{2}} + \frac{x(x-1)(x-\frac{1}{2})}{3!}\delta^3 u_{\frac{1}{2}}$$

$$+ \frac{(x+1)x(x-1)(x-2)}{4!}\mu\delta^4 u_{\frac{1}{2}} + \frac{(x+1)x(x-1)(x-2)(x-\frac{1}{2})}{5!}\delta^5 u_{\frac{1}{2}} + \dots$$

6.11. To deduce Everett's formula assume

$$u_x = A_0 u_0 + A_2\delta^2 u_0 + A_4\delta^4 u_0 + \dots\dots + A_{2r}\delta^{2r}u_0 + \dots\dots$$

$$+ A_1 u_1 + A_3\delta^2 u_1 + A_5\delta^4 u_1 + \dots\dots + A_{2r+1}\delta^{2r}u_1 + \dots\dots$$

Substituting the values of the δ's we have

$$E^x = (A_0 + A_1) + A_1\Delta + (A_2 + A_3)\frac{\Delta^2}{E} + A_3\frac{\Delta^3}{E} + \dots\dots$$

$$+ A_{2r-1}\frac{\Delta^{2r-1}}{E^{r-1}} + (A_{2r} + A_{2r+1})\frac{\Delta^{2r}}{E^r} + \dots\dots$$

Hence multiplying by E^{r-1} and equating the coefficient of Δ^{2r-1} we have

$$A_{2r-1}=\frac{(x+r-1)(x+r-2)\ldots\ldots(x-r+1)}{(2r-1)!}=\binom{x+r-1}{2r-1}.$$

Similarly

$$A_{2r-1}+A_{2r}+A_{2r+1}=\frac{(x+r)(x+r-1)\ldots\ldots(x-r+1)}{(2r)!}=\binom{x+r}{2r},$$

whence
$$A_{2r}=\frac{(x+r-1)(x+r-2)\ldots\ldots(x-r)(1-x+r)}{(2r+1)!}$$

$$=\frac{(y+r)(y+r-1)\ldots\ldots(y-r)}{(2r+1)!},\quad \text{if } y=1-x.$$

Thus

$$u_x=yu_0+\binom{y+1}{3}\delta^2u_0+\binom{y+2}{5}\delta^4u_0+\ldots+\binom{y+r}{2r+1}\delta^{2r}u_0+\ldots$$

$$+xu_1+\binom{x+1}{3}\delta^2u_1+\binom{x+2}{5}\delta^4u_1+\ldots+\binom{x+r}{2r+1}\delta^{2r}u_1+\ldots$$

Other formulæ are also derived in a similar manner.

6.12. Relative Advantages of Different Formulae. Central difference formulae are generally more accurate than Newton's advancing difference formula. The greater accuracy of central difference formulae results from placing x in the central interval of the series of given values. The central difference coefficients are as a rule smaller and more rapidly convergent than those in the advancing difference formula, and by a suitable choice of origin, the arithmetical work may be reduced to a minimum. Further the coefficients of the principal central difference formulae are all tabulated and can be easily applied.

Advancing difference formula is useful in interpolating near the beginning or the end of the given series of values, while central difference formulae are very advantageous in interpolating in the central interval. The selection is a matter of necessity rather than of preference. Successive approximation to the required result may be obtained by the inclusion of more extensive data on both sides while the basic central difference formula remains unchanged, but in the case of advancing differences a fresh set of leading differences and a fresh set of coefficients will be necessary. This is one of the principal advantages of central difference formulae.* When additional data are taken on one side only, advancing difference formula is to be used.

For the bisection of an interval Bessel's form is convenient since the alternate terms are zero. Everett's formula is very useful for subdivision of intervals in the central position (see §6.8). For values of x in the neighbourhood of one half Bessel's form gives more accurate results. When x is quite small Stirling's formula will be found more convenient. See also §6.16.

6.13. The Lozenge Diagram. Many interpolation formulae can be easily obtained with the help of Fraser's hexagon diagram.†

Since
$$\binom{x+r+1}{n+1}=\binom{x+r}{n+1}+\binom{x+r}{n},$$

we have

$$(1) \qquad \binom{x+r}{n}=\binom{x+r+1}{n+1}-\binom{x+r}{n+1}.$$

It is evident that giving to r and n consecutive integral values a table can be obtained akin to the difference table obtained from

$$(2) \qquad \Delta^{n+1}u_{-r-1}=\Delta^{n}u_{-r}-\Delta^{n}u_{-r-1}.$$

Putting the three elements of (2) in the usual tabular form and the elements of (1) in a reverse tabular form, and combining them together, a hexagon is obtained having at its vertices the elements of (1) and (2).

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* For a detailed investigation into the relative accuracy of central and advancing difference formulae, see **W. F. Sheppard**—*Proc. Lond. Math. Soc.*, 4 (1906), pp. 320-41 and 10 (1911), pp. 139-72; **D. C. Fraser**—*Journ. Inst. Act.*, 50 (1916), pp. 25-27; **G. J. Lidstone**—*Trans Fac. Act.*, 9 (1923), pp. 246-57.

† **D. C. Fraser**—*Journ. Inst. Act.*, 43 (1909), pp. 235 *et seq.*

$$\begin{tikzcd}
 & \Delta^n u_{-r-1} \arrow[r, dash] & \binom{x+r+1}{n+1} \arrow[dr, dash] & \\
\binom{x+r}{n} \arrow[ur, dash] \arrow[dr, dash] & & & \Delta^{n+1} u_{-r-1}. \\
 & \Delta^n u_{-r} \arrow[r, dash] & \binom{x+r}{n+1} \arrow[ur, dash] &
\end{tikzcd}$$

This is known as *Fraser's Hexagon Diagram*. As it is in the form of a lozenge it is also called a *Lozenge Diagram*.

From (1) and (2) we have

$$\binom{x+r}{n}\left\{\Delta^n u_{-r} - \Delta^n u_{-r-1}\right\} = \left\{\binom{x+r+1}{n+1} - \binom{x+r}{n+1}\right\}\Delta^{n+1} u_{-r-1}$$

$$(3) \quad \text{or} \quad \binom{x+r}{n}\Delta^n u_{-r-1} + \binom{x+r+1}{n+1}\Delta^{n+1} u_{-r-1}$$

$$= \binom{x+r}{n}\Delta^n u_{-r} + \binom{x+r}{n+1}\Delta^{n+1} u_{-r-1}.$$

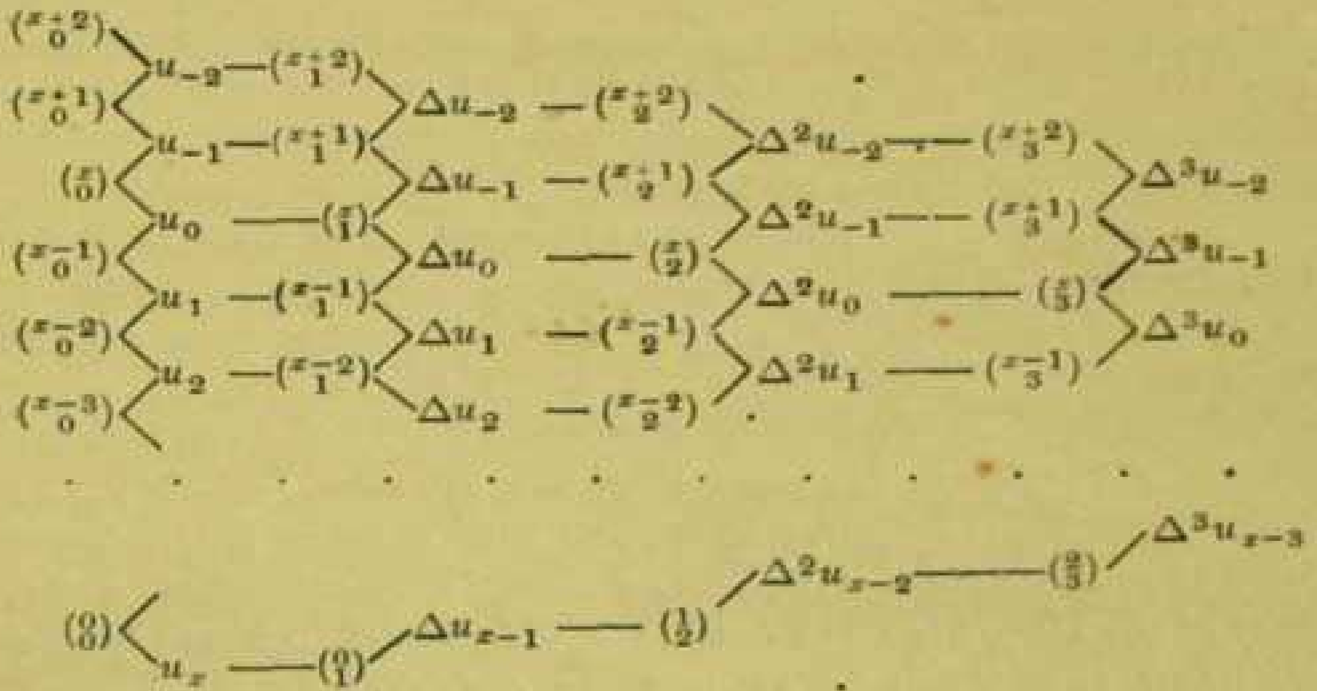
Therefore if we assume that

(*i*) the oblique lines denote the product of the quantities they connect and horizontal lines addition ;

(*ii*) a line taken in a right-hand direction gives the product a positive sign, and in the left-hand direction a negative sign ;

it easily follows from (3) that in travelling from the left-hand vertex to the right-hand vertex of the lozenge, the sum of the elements lying along the upper route is equal to the sum of the elements lying along the lower route. Hence the sum of the operations in travelling round any closed circuit is zero, or simply the *circuit of a lozenge is zero*.

6.14. Placing in contiguity consecutive lozenges corresponding to consecutive integral values of r and n, we get the following lozenge diagram :


$\binom{x+2}{0}$
u_{-2} — $\binom{x+2}{1}$
$\binom{x+1}{0}$
Δu_{-2} — $\binom{x+2}{2}$
u_{-1} — $\binom{x+1}{1}$
$\Delta^2 u_{-2}$ — $\binom{x+2}{3}$
$\binom{x}{0}$
Δu_{-1} — $\binom{x+1}{2}$
$\Delta^3 u_{-2}$
u_0 — $\binom{x}{1}$
$\Delta^2 u_{-1}$ — $\binom{x+1}{3}$
$\binom{x-1}{0}$
Δu_0 — $\binom{x}{2}$
$\Delta^3 u_{-1}$
u_1 — $\binom{x-1}{1}$
$\Delta^2 u_0$ — $\binom{x}{3}$
$\binom{x-2}{0}$
Δu_1 — $\binom{x-1}{2}$
$\Delta^3 u_0$
u_2 — $\binom{x-2}{1}$
$\Delta^2 u_1$ — $\binom{x-1}{3}$
$\binom{x-3}{0}$
Δu_2 — $\binom{x-2}{2}$
$\Delta^3 u_{x-3}$
$\Delta^2 u_{x-2}$ — $\binom{2}{3}$
$\binom{0}{0}$
Δu_{x-1} — $\binom{1}{2}$
u_x — $\binom{0}{1}$

By convention we have $\binom{x+r}{0}=1$ for $r=0, \pm 1, \pm 2, \ldots\ldots$ Consider any group of lozenges. The circuit of each lozenge is zero and therefore the sum of the circuits of all the lozenges will be zero. The operations represented by the internal lines will cancel out, since each is taken once positively and once negatively. There will remain only the external lines of the lozenges and thus *the circuit of any group of lozenges is zero.*

It follows therefore that in passing from any chosen coefficient to any other chosen difference, the result is identically the same whatever route be taken. Thus the following expressions give the same result:

$$u_0+\binom{x}{1}\Delta u_0+\binom{x}{2}\Delta^2 u_0+\binom{x}{3}\Delta^3 u_{-1}$$
$$u_0+\binom{x}{1}\Delta u_0+\binom{x}{2}\Delta^2 u_{-1}+\binom{x+1}{3}\Delta^3 u_{-1}$$
$$u_0+\binom{x}{1}\Delta u_{-1}+\binom{x+1}{2}\Delta^2 u_{-1}+\binom{x+1}{3}\Delta^3 u_{-1}$$
$$u_1+\binom{x-1}{1}\Delta u_0+\binom{x}{2}\Delta^2 u_0+\binom{x}{3}\Delta^3 u_{-1}$$
$$u_1+\binom{x-1}{1}\Delta u_0+\binom{x}{2}\Delta^2 u_{-1}+\binom{x+1}{3}\Delta^3 u_{-1}$$
$$u_1+\binom{x-1}{1}\Delta u_1+\binom{x-1}{2}\Delta^2 u_0+\binom{x}{3}\Delta^3 u_{-1}$$
$$u_{-1}+\binom{x+1}{1}\Delta u_{-1}+\binom{x+1}{2}\Delta^2 u_{-1}+\binom{x+1}{3}\Delta^3 u_{-1}.$$

6.15. Suppose the diagram extends to the column of zero differences, and consider the entire group of lozenges. The zigzag lines on the left-hand side will cancel one another in pairs leaving the two extreme lines, and those on the right-hand side are all zeroes. Since the circuit of the entire group is zero it follows that the upper route from left to right is equivalent to the lower route in the same sense.

We can add to and take away any number of lozenges from the diagram. Hence every route across the diagram, however twisted or circuitous, if it ends at zero difference, gives the same result.

Take now the lowest route through u_x. The coefficient of u_x is 1 and following the route along Δu_{x-1}, $\Delta^2 u_{x-2}$, $\Delta^3 u_{x-3} \ldots$ it is found that the coefficient of each difference is zero. Thus the entire route through u_x gives u_x as the result. Hence *the sum of the elements along any route proceeding from any entry u_r whatever to the line of zero difference is equal to u_x.*

Thus

$$u_x = u_0 + x\Delta u_0 + \binom{x}{2}\Delta^2 u_{-1} + \binom{x+1}{3}\Delta^3 u_{-1} + \binom{x+1}{4}\Delta^4 u_{-2} + \ldots$$

which is Gauss' formula. Again

$$u_x = u_0 + x\Delta u_{-1} + \binom{x+1}{2}\Delta^2 u_{-1} + \binom{x+1}{3}\Delta^3 u_{-2} + \binom{x+2}{4}\Delta^4 u_{-2} + \ldots$$

and taking the mean of the above routes we get the Newton-Stirling formula.

6.16. By a slight modification in the lozenge diagram, formulae involving mean differences can be directly obtained.

Instead of taking the product of the elements connected by oblique lines, take the product of the element at a vertex and the mean of the elements at the end of the two oblique lines meeting at the vertex. Thus to obtain the Newton-Stirling formula which employs differences lying on the horizontal line through u_0, the route begins with u_0 and coming horizontally to $\binom{x}{1}$ multiply it by the mean of Δu_{-1} and Δu_0, and proceeding horizontally to $\Delta^2 u_{-1}$ multiply it by the mean of $\binom{x+1}{2}$ and $\binom{x}{2}$ and so on. Thus

$$u_x = u_0 + \binom{x}{1}\frac{\Delta u_{-} + \Delta u_0}{2} + \frac{1}{2}\left\{\binom{x+1}{2} + \binom{x}{2}\right\}\Delta^2 u_{-1}$$
$$+ \binom{x+1}{3}\frac{\Delta^3 u_{-2} + \Delta^3 u_{-1}}{2} + \ldots$$

or in central difference notation

$$u_x = u_0 + x\mu\delta u_0 + \tfrac{1}{2}x^2\delta^2 u_0 + \tfrac{1}{6}x(x^2-1)\mu\delta^3 u_0 + \dots$$

Newton-Bessel formula is similarly obtained by following the horizontal route through Δu_0.

With regard to these formulae it may be generally remarked:

(*i*) Formulae which proceed to constant differences are exact, and are true for all values of x whether integral or fractional.

(*ii*) Formulae which stop short of constant differences are approximations.

(*iii*) Approximate formulae which terminate with the same difference are identically equal.

(*iv*) Approximate formulae which terminate with distinct differences of the same order are not identical. The difference between them is expressed by the chain of lines necessary to complete the circuit. Thus the difference of two routes ending with $\Delta^3 u_0$ and $\Delta^3 u_{-3}$ respectively, is equal to

$$\begin{aligned}\tbinom{x+2}{3}(\Delta^3 u_{-2} - \Delta^3 u_{-3}) + \tbinom{x+1}{3}(\Delta^3 u_{-1} - \Delta^3 u_{-2}) + \tbinom{x}{3}(\Delta^3 u_0 - \Delta^3 u_{-1})\\ = \tbinom{x+2}{3}\Delta^4 u_{-3} + \tbinom{x+1}{3}\Delta^4 u_{-2} + \tbinom{x}{3}\Delta^4 u_{-1}\\ = \text{a series of horizontal diagonals.}\end{aligned}$$

Mr. Fraser has shown that formulae in divided differences can similarly be obtained by use of the hexagon diagram.*

6.17. Central Factorial Expression. Product of the form

$$x(x + \tfrac{1}{2}n - 1)(x + \tfrac{1}{2}n - 2)\dots\dots(x - \tfrac{1}{2}n + 1)$$

is called the *Central Factorial* and is represented by the symbol $x^{[n]}$. For $n=0$, $x^{[0]}=1$; and for $n>0$, $x^{[n]}=0$ when $x=0$.

It can be put as

(1) $$x^{[n]} = x(x + \tfrac{1}{2}n - 1)^{(n-1)},$$

---

* **D. C. Fraser**—*Newton and Interpolation*, a Memorial volume published by the Mathematical Association, 1927.

The Central Factorials of even and odd order may be written as

(2) $$x^{[2m]} = x^2(x^2-1)(x^2-4)\ldots\ldots\{x^2-(m-1)^2\},$$

$$x^{[2m+1]} = x(x^2-\tfrac{1}{4})(x^2-\tfrac{9}{4})\ldots\ldots\{x^2-\tfrac{1}{4}(2m-1)^2\},$$

which shows that $x^{[2m]}$ is an even function and $x^{[2m+1]}$ an odd function of x.

Also $$x^{[n]-1} = \frac{x^{[n]}}{x} = (x+\tfrac{1}{2}n-1)^{(n-1)}.$$

Taking the central difference of (1) we have

$$\delta x^{[n]} = \left(x+\frac{1}{2}\right)\left(x+\frac{n-1}{2}\right)^{(n-1)} - \left(x-\frac{1}{2}\right)\left(x+\frac{n-1}{2}-1\right)^{(n-1)}$$

$$= \left(x+\frac{n-1}{2}-1\right)^{(n-2)}\left\{\left(x+\frac{1}{2}\right)\left(x+\frac{n-1}{2}\right) - \left(x-\frac{1}{2}\right)\left(x-\frac{n-1}{2}\right)\right\}$$

$$= \left(x+\frac{n-1}{2}-1\right)^{(n-2)}.nx$$

$$= nx^{[n-1]}.$$

Similarly $$\delta^2 x^{[n]} = n(n-1)x^{[n-2]},$$

(3) $$\delta^r x^{[n]} = n^{(r)}x^{[n-r]},$$

$$\delta^n x^{[n]} = n!.$$

6.18. Expansion of a Polynomial in Central Factorials. Let $\phi(x)$ be a polynomial of degree n. To express it in Central Factorials assume

(1) $$\phi(x) = a_0 + a_1x^{[1]} + a_2x^{[2]} + \ldots + a_rx^{[r]} + \ldots + a_nx^{[n]}$$

where the coefficients $a_0, a_1, a_2\ldots\ldots$ are to be determined.

Taking the central differences of (1) we have

$$\delta\phi(x) = a_1 + 2a_2x^{[1]} + 3a_3x^{[2]} + \ldots + ra_rx^{[r-1]} + \ldots + na_nx^{[n-1]}$$

$$\delta^2\phi(x) = 2a_2 + 3.2.a_3x^{[1]} + \ldots + r(r-1)a_rx^{[r-2]} \ldots + n(n-1)a_nx^{[n-2]}$$

$$\delta^r\phi(x) = r^{(r)}a_r + (r+1)^{(r)}a_{r+1}x^{[1]} + \ldots + n^{(r)}a_nx^{[n-r]}$$

$$\delta^n\phi(x) = n^{(n)}a_n.$$

Hence putting $x=0$, we have

$$\phi(0)=a_0 \text{ and } \delta^r\phi(0)=a_r . r!.$$

$$(2) \quad \therefore \ \phi(x)=\phi(0)+\frac{x^{[1]}}{1!}\delta\phi(0)+\frac{x^{[2]}}{2!}\delta^2\phi(0)+\dots$$
$$+\frac{x^{[r]}}{r!}\delta^r\phi(0)+\dots+\frac{x^{[n]}}{n!}\delta^n\phi(0)$$
$$=\sum_{r=0}^{n}\frac{x^{[r]}}{r!}\delta^r\phi(0).$$

In particular, putting $\phi(x)=x^n$, we get

$$(3) \qquad x^n=\sum_{r=0}^{n}\frac{x^{[r]}}{r!}\delta^r 0^n.$$

The values of $\delta^m 0^n$ are known as *Central Differences of Nothing*. They are obtained thus:

$$\delta^m\phi(x)=\left(E^{\frac{1}{2}}-E^{-\frac{1}{2}}\right)^m\phi(x)$$
$$=\phi\left(x+\frac{m}{2}\right)-m\phi\left(x+\frac{m}{2}-1\right)+\dots+(-1)^m\phi\left(x-\frac{m}{2}\right).$$

$$(4) \quad \therefore \ \delta^m 0^n=\left(\frac{m}{2}\right)^n-\binom{m}{1}\left(\frac{m}{2}-1\right)^n+\binom{m}{2}\left(\frac{m}{2}-2\right)^n-\dots$$
$$+(-1)^m\left(-\frac{m}{2}\right)^n.$$
$$=\sum_{r=0}^{m}(-1)^r\binom{m}{r}\left(\frac{m}{2}-r\right)^n.$$

Taking together the first term and the last term, the second and the last but one term, etc., we find

$$\delta^m 0^n=0, \quad \text{when } m+n \text{ is odd,}$$

$$\delta^m 0^n=2\sum_{r=0}^{r<\frac{m}{2}}(-1)^r\binom{m}{r}\left(\frac{m}{2}-r\right)^n, \text{ when } m+n \text{ is even.}$$

6.19. Expression of a Central Factorial as a Polynomial.

The central factorial $x^{[n]}$ can similarly be expressed in powers of x by assuming

$$x^{[n]} = \sum_{r=0}^{n} a_r x^r$$

and then differentiating it with respect to x. Thus we get

$$x^{[n]} = \sum_{r=0}^{n} x^r \frac{D^r 0^{[n]}}{r!}. \tag{1}$$

Now from §6.17(2) it is seen that the expansion of $x^{[2m]}$ contains only even powers of x and the expansion of $x^{[2m+1]}$ only odd powers. Therefore $D^r 0^{[n]} = 0$, if $n+r$ is odd. Thus

$$x^{[2m]} = \sum_{r=1}^{m} x^{2r} \frac{D^{2r} 0^{[2m]}}{(2r)!},$$

$$x^{[2m+1]} = \sum_{r=0}^{m} x^{2r+1} \frac{D^{2r+1} 0^{[2m+1]}}{(2r+1)!}. \tag{2}$$

The coefficients in the expansion of $x^{[n]}$ have alternating signs. $D^{2r}0^{[2m]}$ and $D^{2r+1}0^{[2m+1]}$ have both the signs $(-1)^{m+r}$.

The coefficients can be calculated thus:

Since $$x^{[2m+2]} = (x^2 - m^2) x^{[2m]},$$

we have by Leibnitz's theorem

$$\begin{aligned} D^{2r} x^{[2m+2]} &= (x^2 - m^2) D^{2r} x^{[2m]} + \binom{2r}{1} D^{2r-1} x^{[2m]} . D(x^2 - m^2) \\ &\qquad + \binom{2r}{2} D^{2r-2} x^{[2m]} . D^2 (x^2 - m^2) \\ &= (x^2 - m^2) D^{2r} x^{[2m]} + 4rx D^{2r-1} x^{[2m]} + (2r)^{(2)} D^{2r-2} x^{[2m]}. \end{aligned}$$

Putting $x=0$, we have the recurrence formula

$$D^{2r} 0^{[2m+2]} = 2r(2r-1) D^{2r-2} 0^{[2m]} - m^2 D^{2r} 0^{[2m]}, \tag{3}$$

with the initial values

$$\frac{D^2 0^{[2m]}}{2!} = (-1)^{m-1} \{(m-1)!\}^2, \quad \frac{D^{2m} 0^{[2m]}}{(2m)!} = 1. \tag{4}$$

In a similar manner from $x^{[2m+1]}$ we have the recurrence formula

$$(5)\quad D^{2r+1}0^{[2m+1]} = (2r+1)2rD^{2r-1}0^{[2m-1]} - \tfrac{1}{4}(2m-1)^2 D^{2r+1}0^{[2m-1]},$$

with the initial values

$$(6)\quad D0^{[2m-1]} = (-1)^{m-1}\left\{\frac{1.3.\ldots.(2m-3)}{2^{m-1}}\right\}^2, \quad \frac{D^{2m-1}0^{[2m-1]}}{(2m-1)\,!} = 1.$$

Putting $x=1$ in (2), we get

$$(7)\quad \sum_{r=1}^{m} \frac{D^{2r}0^{[2m]}}{(2r)\,!} = 0, \quad m>1$$

$$\sum_{r=0}^{m} \frac{D^{2r+1}\,0^{[2m+1]}}{(2r+1)\,!} = (-1)^{m-1}\frac{3}{2^{2m}}(9-4)(25-4)\ldots\{(2m-1)^2-4\}.$$

6.20. Derivatives from Newton-Stirling Formula. Arranging the terms of Newton-Stirling formula §6.5(3) in ascending powers of x, we have

$$\begin{aligned}(1)\quad f(a+x\omega) = u_x = u_0 &+ x(\mu\delta u_0 - \tfrac{1}{6}\mu\delta^3 u_0 + \tfrac{1}{30}\mu\delta^5 u_0 - \tfrac{1}{140}\mu\delta^7 u_0 + \ldots)\\ &+ \frac{x^2}{2\,!}(\delta^2 u_0 - \tfrac{1}{12}\delta^4 u_0 + \tfrac{1}{90}\delta^6 u_0 - \tfrac{1}{560}\delta^8 u_0 + \ldots)\\ &+ \frac{x^3}{3\,!}(\mu\delta^3 u_0 - \tfrac{1}{4}\mu\delta^5 u_0 + \tfrac{7}{120}\mu\delta^7 u_0 - \ldots)\\ &+ \frac{x^4}{4\,!}(\delta^4 u_0 - \tfrac{1}{6}\delta^6 u_0 + \tfrac{7}{240}\delta^8 u_0 - \ldots)\\ &+ \frac{x^5}{5\,!}(\mu\delta^5 u_0 - \tfrac{1}{3}\mu\delta^7 u_0 + \ldots)\\ &+ \frac{x^6}{6\,!}(\delta^6 u_0 - \tfrac{1}{4}\delta^8 u_0 + \ldots)\\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

Again by Taylor's theorem we have

$$(2)\quad f(a+x\omega) = f(a) + x\omega f'(a) + \frac{x^2\omega^2}{2\,!}f''(a) + \frac{x^3\omega^3}{3\,!}f'''(a) + \ldots$$

Hence comparing the coefficients of different powers of x in (1) and (2) ge get

$$(3)\quad \omega f'(a)=\mu\delta u_0-\tfrac{1}{6}\mu\delta^3u_0+\tfrac{1}{30}\mu\delta^5u_0-\tfrac{1}{140}\mu\delta^7u_0+\dots$$

$$\omega^2f''(a)=\delta^2u_0-\tfrac{1}{12}\delta^4u_0+\tfrac{1}{90}\delta^6u_0-\tfrac{1}{560}\delta^8u_0+\dots$$

$$\omega^3f'''(a)=\mu\delta^3u_0-\tfrac{1}{4}\mu\delta^5u_0+\tfrac{7}{120}\mu\delta^7u_0-\dots$$

$$\omega^4f''''(a)=\delta^4u_0-\tfrac{1}{6}\delta^6u_0+\tfrac{7}{240}\delta^8u_0-\dots$$

$$\omega^5f^{v}(a)=\mu\delta^5u_0-\tfrac{1}{3}\mu\delta^7u_0+\dots$$

$$\omega^6f^{vi}(a)=\delta^6u_0-\tfrac{1}{4}\delta^8u_0+\dots$$

....

The above set gives the values of derivatives of $f(a)$. To obtain the values of $f^r(a+x\omega)$, formula (1) has to be differentiated r times.

6.21. Derivatives from Newton-Bessel Formula. Arranging Newton-Bessel Formula §6.6(5) in ascending powers of x, we have

$$(1)\quad u_{x+\frac{1}{2}}=\mu u_{\frac{1}{2}}+x(\delta u_{\frac{1}{2}}-\tfrac{1}{24}\delta^3u_{\frac{1}{2}}+\tfrac{3}{640}\delta^5u_{\frac{1}{2}}-\tfrac{5}{7168}\delta^7u_{\frac{1}{2}}+\dots)$$
$$+\frac{x^2}{2!}(\mu\delta^2u_{\frac{1}{2}}-\tfrac{5}{24}\mu\delta^4u_{\frac{1}{2}}+\tfrac{259}{5760}\mu\delta^6u_{\frac{1}{2}}-\dots)$$
$$+\frac{x^3}{3!}(\delta^3u_{\frac{1}{2}}-\tfrac{1}{8}\delta^5u_{\frac{1}{2}}+\tfrac{37}{1920}\delta^7u_{\frac{1}{2}}-\dots)$$
$$+\frac{x^4}{4!}(\mu\delta^4u_{\frac{1}{2}}-\tfrac{7}{24}\mu\delta^6u_{\frac{1}{2}}+\dots)$$
$$+\frac{x^5}{5!}(\delta^5u_{\frac{1}{2}}-\tfrac{5}{24}\delta^7u_{\frac{1}{2}}+\dots)$$
$$+\dots\dots\dots\dots$$

Hence

$$(2)\quad \omega f'(a+\tfrac{1}{2}\omega)=\delta u_{\frac{1}{2}}-\tfrac{1}{24}\delta^3u_{\frac{1}{2}}+\tfrac{3}{640}\delta^5u_{\frac{1}{2}}-\tfrac{5}{7168}\delta^7u_{\frac{1}{2}}+\dots$$

$$\omega^2f''(a+\tfrac{1}{2}\omega)=\mu\delta^2u_{\frac{1}{2}}-\tfrac{5}{24}\mu\delta^4u_{\frac{1}{2}}+\tfrac{259}{5760}\mu\delta^6u_{\frac{1}{2}}-\dots$$

$$\omega^3f'''(a+\tfrac{1}{2}\omega)=\delta^3u_{\frac{1}{2}}-\tfrac{1}{8}\delta^5u_{\frac{1}{2}}+\tfrac{37}{1920}\delta^7u_{\frac{1}{2}}-\dots$$

$$\omega^4f''''(a+\tfrac{1}{2}\omega)=\mu\delta^4u_{\frac{1}{2}}-\tfrac{7}{24}\mu\delta^6u_{\frac{1}{2}}+\dots$$

$$\omega^5f^{v}(a+\tfrac{1}{2}\omega)=\delta^5u_{\frac{1}{2}}-\tfrac{5}{24}\delta^7u_{\frac{1}{2}}+\dots$$

These determine the derivatives at points *midway* between the tabular values of the function.

Derivatives of $f\{a+(x+\frac{1}{2})\omega\}$ are obtained by differentiating (1). Derivatives obtained from Newton-Bessel formula §6.6(4) will be

$$(3)\qquad \omega f'(a)=\delta u_{\frac{1}{2}}-\tfrac{1}{2}\mu\delta^2 u_{\frac{1}{2}}+\tfrac{1}{12}\delta^3 u_{\frac{1}{2}}+\tfrac{1}{12}\mu\delta^4 u_{\frac{1}{2}}-\tfrac{1}{120}\delta^5 u_{\frac{1}{2}}-\dots$$

$$\omega^2 f''(a)=\mu\delta^2 u_{\frac{1}{2}}-\tfrac{1}{2}\delta^3 u_{\frac{1}{2}}-\tfrac{1}{12}\mu\delta^4 u_{\frac{1}{2}}+\tfrac{1}{24}\delta^5 u_{\frac{1}{2}}+\dots$$

$$\omega^3 f'''(a)=\delta^3 u_{\frac{1}{2}}-\tfrac{1}{2}\mu\delta^4 u_{\frac{1}{2}}-\dots$$

$$\omega^4 f''''(a)=\mu\delta^4 u_{\frac{1}{2}}-\tfrac{1}{2}\delta^5 u_{\frac{1}{2}}-\dots$$

$$\omega^5 f^{v}(a)=\delta^5 u_{\frac{1}{2}}-\dots$$

……………………

Example 1. Values of $f(\theta)=\cos\theta$ are given for $\theta=20°$, 30°, 40°, 50°, 60° and 70°. Find the value of $f'(\theta)$ for $\theta=45°$.

Forming the difference table, we have

20°	·93969					
		−·07366				
30°	·86603		−·02633			
		−·09999		307		
40°	·76604		−2326		65	
		−·12325		372		−2
50°	·64279		−1954		63	
		−·14279		435		
60°	·50000		−1519			
		−·15798				
70°	·34202					

Here $\omega=10°$, take $a=40°$. As the interval is halved, we take the derivative from Newton-Bessel formula. By the first formula of §6.21(2) we get

$$\omega f'(45°)=-{\cdot}12325-\frac{1}{24}\times{\cdot}00372-\frac{3}{640}\times{\cdot}00002$$

$$=-{\cdot}123405\dots$$

Now $$\omega = 10^\circ = \frac{\pi}{18} = \cdot 174533.$$

$\therefore$ $f'(45^\circ) = -\cdot 7071$ correct to four places of decimals.

Example 2. Given the values of $f(n)$ for $n = 0, 2, 4, 6, 8, 10$. Find the value of $f''(n)$ for $n = 4{\cdot}8$.

The table is formed below:

0	·6427876				
		·0263430			
2	·6691306		−8152		
		255278		−312	
4	·6946584		−8464		12
		246814		−300	
6	·7193398		−8764		10
		238050		−290	
8	·7431448		−9054		
		228996			
10	·7660444				

Here both Newton-Bessel and Newton-Stirling formulae are applicable, and both give the same result.

Here $\omega = 2$ and $x = \cdot 4$. Take $a = 4$. Differentiating Newton-Stirling formula §6.20(1) twice we have

$$\omega^2 f''(4{\cdot}8) = \delta^2 u_0 - \frac{1}{12}\delta^4 u_0 + x\mu\delta^3 u_0 + \frac{x^2}{2\,!}\delta^4 u_0$$

$$= -\cdot 0008464 - \cdot 0000001 - \cdot 00001224 + \cdot 000000096$$

$$= -\cdot 000858644$$

$\therefore$ $f''(4{\cdot}8) = -\cdot 000214661.$

Differentiating Newton-Bessel formula twice we have

$$\omega^2 f''(4{\cdot}8) = \mu\delta^2 u_{\frac{1}{2}} + \left(x - \frac{1}{2}\right)\delta^3 u_{\frac{1}{2}} + \left(\frac{x^2}{2} - \frac{x}{2} - \frac{1}{12}\right)\mu\delta^4 u_{\frac{1}{2}}$$

$$= -\cdot 0008614 + \cdot 0000030 - \cdot 0000002236$$

$$= -\cdot 000858624$$

$\therefore$ $f''(4{\cdot}8) = -\cdot 000214656.$

Thus the two results are same correct to 8 places of decimals.

6.22. Numerical Integration by Stirling's Formula. We have by §6.5(2),

$$u_x = u_0 + x\mu\delta u_0 + \frac{x^2}{2!}\delta^2 u_0 + \frac{x(x^2-1)}{3!}\mu\delta^3 u_0 + \frac{x^2(x^2-1)}{4!}\delta^4 u_0$$

$$+ \frac{x(x^2-1)(x^2-4)}{5!}\mu\delta^5 u_0 + \frac{x^2(x^2-1)(x^2-4)}{6!}\delta^6 u_0 + \dots$$

Integrating this, we get

$$(1)\quad \int u_x dx = xu_0 + \frac{x^2}{2}\mu\delta u_0 + \frac{x^3}{6}\delta^2 u_0 + \frac{1}{12}\left(\frac{x^4}{2} - x^2\right)\mu\delta^3 u_0$$

$$+ \frac{1}{24}\left(\frac{x^5}{5} - \frac{x^3}{3}\right)\delta^4 u_0 + \frac{1}{120}\left(\frac{x^6}{6} - \frac{5}{4}x^4 + 2x^2\right)\mu\delta^5 u_0$$

$$+ \frac{1}{720}\left(\frac{x^7}{7} - x^5 + \frac{4}{3}x^3\right)\delta^6 u_0 + \dots + C,$$

where C is the constant of integration. Taking (1) between the limits $x = -\frac{1}{2}$ and $x = \frac{1}{2}$ we have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} u_x dx = u_0 + \frac{1}{24}\delta^2 u_0 - \frac{17}{5760}\delta^4 u_0 + \frac{367}{967680}\delta^6 u_0 - \dots$$

In the same way

$$\int_{\frac{1}{2}}^{\frac{3}{2}} u_x dx = u_1 + \frac{1}{24}\delta^2 u_1 - \frac{17}{5760}\delta^4 u_1 + \frac{367}{967680}\delta^6 u_1 - \dots$$

. . . .

$$\int_{r-\frac{1}{2}}^{r+\frac{1}{2}} u_x dx = u_r + \frac{1}{24}\delta^2 u_r - \frac{17}{5760}\delta^4 u_r + \frac{367}{967680}\delta^6 u_r - \dots$$

Adding these together

$$\int_{-\frac{1}{2}}^{r+\frac{1}{2}} u_x dx = \sum_0^r u_r + \frac{1}{24}\sum_0^r \delta^2 u_r - \frac{17}{5760}\sum_0^r \delta^4 u_r + \frac{367}{967680}\sum_0^r \delta^6 u_r - \dots$$

Now $\delta^2 u_0 + \delta^2 u_1 + \delta^2 u_2 + \dots + \delta^2 u_r = \delta u_{r+\frac{1}{2}} - \delta u_{-\frac{1}{2}}$

$$\delta^{2m} u_0 + \delta^{2m} u_1 + \delta^{2m} u_2 + \dots + \delta^{2m} u_r = \delta^{2m-1} u_{r+\frac{1}{2}} - \delta^{2m-1} u_{-\frac{1}{2}}.$$

Hence substituting these values, we get

$$(2)\quad \int_{-\frac{1}{2}}^{r+\frac{1}{2}} u_x\,dx = u_0 + u_1 + \dots + u_r + \frac{1}{24}\left(\delta u_{r+\frac{1}{2}} - \delta u_{-\frac{1}{2}}\right) - \frac{17}{5760}\left(\delta^3 u_{r+\frac{1}{2}} - \delta^3 u_{-\frac{1}{2}}\right) + \frac{367}{967680}\left(\delta^5 u_{r+\frac{1}{2}} - \delta^5 u_{-\frac{1}{2}}\right) - \dots$$

$$(3)\quad \therefore \frac{1}{\omega}\int_{a-\frac{1}{2}\omega}^{a+r\omega+\frac{1}{2}\omega} f(x)\,dx = \int_{-\frac{1}{2}}^{r+\frac{1}{2}} f(a+x\omega)\,dx$$

$$= f(a) + f(a+\omega) + \dots + f(a+r\omega)$$

$$+ \tfrac{1}{24}\{\delta f(a+r\omega+\tfrac{1}{2}\omega) - \delta f(a-\tfrac{1}{2}\omega)\}$$

$$- \tfrac{17}{5760}\{\delta^3 f(a+r\omega+\tfrac{1}{2}\omega) - \delta^3 f(a-\tfrac{1}{2}\omega)\}$$

$$+ \tfrac{367}{967680}\{\delta^5 f(a+r\omega+\tfrac{1}{2}\omega) - \delta^5 f(a-\tfrac{1}{2}\omega)\}$$

$$- \dots\dots\dots$$

Example. Find the value of

$$\int_{30^\circ}^{45^\circ} \cot\theta\, d\theta.$$

from the following values of cot θ for $\theta = 22°30'$, $27°30'$,...

Forming the difference table we have

θ	cot θ			
22°30′	2·4140			
		−·4930		
27°30′	1·9210		·1417	
		−·3513		−·0569
32°30′	1·5697		·0848	
		−·2665		−·0302
37°30′	1·3032		·0546	
		−·2119		−·0177
42°30′	1·0913		·0369	
		−·1750		−·0109
47°30′	0·9163		·0260	
		−·1490		
52°30′	0·7673			

Here $\omega=5^\circ=\frac{\pi}{36}$. Take $a=32^\circ 30'$ and $r=2$. Then by (3) we have

$$\frac{1}{\omega}\int_{30^\circ}^{45^\circ}\cot\theta d\theta=1\cdot5697+1\cdot3032+1\cdot0913+\tfrac{1}{24}(-\cdot1750+\cdot3513)$$

$$-\tfrac{17}{5760}(-\cdot0109+\cdot0569)$$

$$=3\cdot9642+\tfrac{1}{24}\times\cdot1763-\tfrac{17}{5760}\times\cdot0460$$

$$=3\cdot9714.$$

As the coefficients converge rapidly, fifth differences have been neglected here.

$\therefore\quad \int_{30^\circ}^{45^\circ}\cot\theta d\theta=3\cdot9714\omega=\cdot3465$, correct to four places of decimals.

6.23. Numerical Integration by Bessel's Formula. Integrating Bessel's formula §6.6(3)

$$u_x=\mu u_{\frac{1}{2}}+(x-\tfrac{1}{2})\delta u_{\frac{1}{2}}+\frac{x(x-1)}{2!}\mu\delta^2 u_{\frac{1}{2}}+\frac{x(x-1)(x-\frac{1}{2})}{3!}\delta^3 u_{\frac{1}{2}}$$

$$+\frac{(x+1)x(x-1)(x-2)}{4!}\mu\delta^4 u_{\frac{1}{2}}+\frac{(x+1)x(x-1)(x-2)(x-\frac{1}{2})}{5!}\delta^5 u_{\frac{1}{2}}$$

$$+\frac{(x+2)(x+1)x(x-1)(x-2)(x-3)}{6!}\mu\delta^6 u_{\frac{1}{2}}+\dots$$

between the limits 0 and 1, we have

$$\int_0^1 u_x dx=\mu u_{\frac{1}{2}}-\tfrac{1}{12}\mu\delta^2 u_{\frac{1}{2}}+\tfrac{11}{720}\mu\delta^4 u_{\frac{1}{2}}-\tfrac{191}{60480}\mu\delta^6 u_{\frac{1}{2}}+\dots$$

In the same way

$$\int_1^2 u_x dx=\mu u_{\frac{3}{2}}-\tfrac{1}{12}\mu\delta^2 u_{\frac{3}{2}}+\tfrac{11}{720}\mu\delta^4 u_{\frac{3}{2}}-\tfrac{191}{60480}\mu\delta^6 u_{\frac{3}{2}}+\dots$$

and generally

$$\int_{r-1}^{r} u_x dx=\mu u_{r-\frac{1}{2}}-\tfrac{1}{12}\mu\delta^2 u_{r-\frac{1}{2}}+\tfrac{11}{720}\mu\delta^4 u_{r-\frac{1}{2}}-\tfrac{191}{60480}\mu\delta^6 u_{r-\frac{1}{2}}+\dots$$

Again since

$$\mu\delta^{2m}u_{\frac{1}{2}}+\mu\delta^{2m}u_{\frac{3}{2}}+\ldots+\mu\delta^{2m}u_{r-\frac{1}{2}}=\mu\delta^{2m-1}u_r-\mu\delta^{2m-1}u_0,$$

we have by addition

$$\begin{aligned}\int_0^r u_x dx=&\tfrac{1}{2}u_0+u_1+u_2+\ldots+u_{r-1}+\tfrac{1}{2}u_r\\ &-\tfrac{1}{12}(\mu\delta u_r-\mu\delta u_0)+\tfrac{11}{720}(\mu\delta^3 u_r-\mu\delta^3 u_0)\\ &-\tfrac{191}{60480}(\mu\delta^5 u_r-\mu\delta^5 u_0)+\ldots\end{aligned}$$

$$\therefore\qquad \frac{1}{\omega}\int_a^{a+r\omega}f(x)dx=\int_0^r f(a+x\omega)dx$$

$$\begin{aligned}&=\tfrac{1}{2}f(a)+f(a+\omega)+\ldots+f(a+\overline{r-1}\omega)+\tfrac{1}{2}f(a+r\omega)\\ &-\tfrac{1}{12}\{\mu\delta f(a+r\omega)-\mu\delta f(a)\}+\tfrac{11}{720}\{\mu\delta^3 f(a+r\omega)-\mu\delta^3 f(a)\}\\ &-\tfrac{191}{60480}\{\mu\delta^5 f(a+r\omega)-\mu\delta^5 f(a)\}+\ldots\end{aligned}$$

This formula will be very useful for the calculation of perturbations.

Example. Calculate the value of $\int_{28^\circ}^{36^\circ}\cos\theta d\theta$ from the data given on p. 105.

Here $\omega=4^\circ=\frac{\pi}{45}={\cdot}069813$. Take $a=28^\circ$ and $r=2$.

From the table on p. 113, we calculate

$\mu\delta\cos 36^\circ={\cdot}041002$, $\mu\delta\cos 28^\circ={\cdot}0327485$

$\mu\delta^3\cos 36^\circ={\cdot}000200$, $\mu\delta^3\cos 28^\circ={\cdot}000159$

$$\begin{aligned}\text{Hence }\frac{1}{4^\circ}\int_{28^\circ}^{36^\circ}\cos\theta d\theta=\tfrac{1}{2}\cos 28^\circ+\cos 32^\circ+\tfrac{1}{2}\cos 36^\circ&=1{\cdot}6940305\\ -\tfrac{1}{12}(\mu\delta\cos 36^\circ-\mu\delta\cos 28^\circ)\qquad&-\ {\cdot}0006878\\ +\tfrac{11}{720}(\mu\delta^3\cos 36^\circ-\mu\delta^3\cos 28^\circ)\qquad&+\ {\cdot}0000006\\ &\overline{\ 1{\cdot}6933433\ }\end{aligned}$$

$$\therefore\qquad \int_{28^\circ}^{36^\circ}\cos\theta d\theta=1{\cdot}6933433\times{\cdot}069813={\cdot}1182173.$$

Examples

1. Prove that

(i) $E^{\frac{1}{2}}u_x = \left(\mu + \frac{1}{2}\delta\right)u_x,$

(ii) $E^{\frac{1}{2}}u_x = \left\{\frac{1}{2}\delta + \left(1 + \frac{\delta^2}{4}\right)^{\frac{1}{2}}\right\}u_x,$

(iii) $\mu^2 u_x = \left(1 + \frac{1}{4}\delta^2\right)u_x.$

(iv) $Du_x = 2\log\left\{\frac{1}{2}\delta + \left(1 + \frac{1}{4}\delta^2\right)^{\frac{1}{2}}\right\}u_x.$

2. Prove that

(i) $\sum_{r=1}^{m}(-1)^{r-1}\{(r-1)\,!\}^2 \frac{\delta^{2r}0^{2m}}{(2r)\,!} = 0, \quad m > 1.$

(ii) $\sum_{r=0}^{m}(-1)^{r}\frac{\{1.3\ldots(2r-1)\}^2}{2^{2r}} \frac{\delta^{2r+1}0^{2m+1}}{(2r+1)\,!} = 0, \quad m \geq 0.$

3. Given $u_0 = 2854$, $u_4 = 3162$, $u_8 = 3544$ and $u_{12} = 3992$, find the value u_5 by Bessel's formula.

4. Given
$\log 310 = 2{\cdot}4913617$
$\log 320 = 2{\cdot}5051500$
$\log 330 = 2{\cdot}5185139$
$\log 340 = 2{\cdot}5314789$
$\log 350 = 2{\cdot}5440680$
$\log 360 = 2{\cdot}5563025$

Find the value of log 3375 by Newton-Bessel formula.

5. Use Newton-Stirling formula to find u_{32} from the data

$u_{20} = 14{\cdot}035$
$u_{25} = 13{\cdot}674$
$u_{30} = 13{\cdot}257$
$u_{35} = 12{\cdot}734$
$u_{40} = 12{\cdot}089$
$u_{45} = 11{\cdot}309$

Explain how you would alter your method if you are required to find all values from u_{20} to u_{45}.

6. Establish the formula

$$u_x = -\tfrac{1}{6}(x^2 - \tfrac{1}{4})(x - \tfrac{3}{2})u_{-\frac{3}{2}} + \tfrac{1}{2}(x - \tfrac{1}{2})(x^2 - \tfrac{9}{4})u_{-\frac{1}{2}}$$
$$- \tfrac{1}{2}(x + \tfrac{1}{2})(x^2 - \tfrac{9}{4})u_{\frac{1}{2}} + \tfrac{1}{6}(x^2 - \tfrac{1}{4})(x^2 + \tfrac{3}{2})u_{\frac{3}{2}}.$$

7. Shew that

$$\tfrac{1}{2}(E^x+E^{-x})=1+\frac{x^2}{2!}\delta^2+\frac{x^2(x^2-1)}{4!}\delta^4+\frac{x^2(x^2-1)(x^2-4)}{6!}\delta^6+\dots$$

$$\tfrac{1}{2}(E^x+E^{-x})=\mu+\frac{x^2-\frac{1}{4}}{2!}\mu\delta^2+\frac{(x^2-\frac{1}{4})(x^2-\frac{9}{4})}{4!}\mu\delta^4+\dots$$

$$\tfrac{1}{2}(E^x-E^{-x})=x\mu\delta+\frac{x(x^2-1)}{3!}\mu\delta^3+\frac{x(x^2-1)(x^2-4)}{5!}\mu\delta^5+\dots$$

$$\tfrac{1}{2}(E^x-E^{-x})=x\delta+\frac{x(x^2-\frac{1}{4})}{3!}\delta^3+\frac{x(x^2-\frac{1}{4})(x^2-\frac{9}{4})}{5!}\delta^5+\dots$$

8. Prove that

$$u_{\frac{1}{2}}=\mu u_{\frac{1}{2}}-\frac{1}{2!2^2}\mu\delta^2u_{\frac{1}{2}}+\frac{3^2}{4!2^4}\mu\delta^4u_{\frac{1}{2}}-\frac{3^2.5^2}{6!2^6}\mu\delta^6u_{\frac{1}{2}}+\dots$$

9. Shew that

$$u_{\frac{n+1}{2}}-u_{\frac{-n+1}{2}}=n\delta u_{\frac{1}{2}}+\frac{n(n^2-1)}{3!2^2}\delta^3u_{\frac{1}{2}}+\frac{n(n^2-1)(n^2-9)}{5!2^4}\delta^5u_{\frac{1}{2}}+\dots$$

10. Find the value of $f'(x)$ and $f'''(x)$ for $x=115$, having given that

x	105	110	115	120	125
$f(x)$	17·53	20·80	24·54	28·83	33·71

11. Shew that the lozenge diagram method really derives all the interpolation formulae by repeated summation by parts, *i.e.*, by the use of the formula

$$u_{x+1}\Delta v_x=\Delta(u_xv_x)-v_x\Delta u_x.$$

12. Prove that

$$x^{2m}=\sum_{r=1}^{m}x^{[2r]}\frac{\delta^{2r}0^{2m}}{(2r)!}$$

and

$$x^{2m+1}=\sum_{r=0}^{m}x^{[2r+1]}\frac{\delta^{2r+1}0^{2m+1}}{(2r+1)!}.$$

13. Prove that $\delta^m 0^n$ is the coefficient of $\frac{1}{n!}x^n$ in the expansion of $\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^m$.

14. Calculate $f'(75)$ and $f''(67)$ from the following data:

		60	4·09434
45	3·80666	65	4·17439
50	3·91202	70	4·24850
55	4·00733	75	4·31749

15. The equation of a curve being $y = x^2\sqrt{2 - \sin x}$, find the area included between the curve, the axis of x and the ordinates

$$x = \tfrac{1}{5}\pi \text{ and } x = \tfrac{2}{7}\pi.$$

16. Evaluate $\int_2^6 x^3 dx$, $\int_2^{11\cdot6} x^3 dx$ by numerical integration.

17. Calculate $\int_{2\cdot5}^{7\cdot5} f(x)dx$ from the following data:

x	$f(x)$	x	$f(x)$	x	$f(x)$
1	1·22751	4	1·47236	7	1·90053
2	1·29480	5	1·58879	8	2·11045
3	1·37552	6	1·72945	9	2·37089

18. Find by Bessel's formula the value of $\int_0^{\frac{\pi}{2}} \theta \sin\theta d\theta$.

19. Find the value of $f'(x)$ and $f'''(x)$ from the data given in Example 2, p. 174.

20. The semi-major axis of an ellipse, $a=1$, and the semi-minor axis, $b=0\cdot8$. Find the length of the elliptic quadrant.

Note.—Take the eccentric angle E as the independent variable; the required length s is given by

$$s = \int_0^{\frac{\pi}{2}} \sqrt{1 - e^2\cos^2 E}\, dE$$

where e is the eccentricity.

21. Find by mechanical quadrature the length of the portion of a cardioid $r = 1 + \cos\theta$ comprised between the initial line and a line through the pole at right angles to the initial line.

CHAPTER VII

RECIPROCAL DIFFERENCES AND OTHER METHODS OF INTERPOLATION

7.1. Reciprocal Differences. Suppose the values of $f(x)$ are given for the arguments $x_0, x_1, x_2, \ldots, x_n$, no two of these being equal. We define Reciprocal Difference* of *the first order* of $f(x)$ for the arguments x_0, x_1, by

$$\rho(x_0x_1)=\frac{x_0-x_1}{f(x_0)-f(x_1)}. \tag{1}$$

Evidently (1) is the reciprocal of the divided difference $f(x_0x_1)$. But the reciprocal difference of *the second order* for three arguments is not obtained by the repetition of the operation denoted by ρ. It is defined by

$$\rho_2(x_0x_1x_2)=\frac{x_0-x_2}{\rho(x_0x_1)-\rho(x_1x_2)}+f(x_1). \tag{2}$$

The operator ρ does not obey the index law or distributive law. Similarly reciprocal difference of *the third order* for four arguments is defined by

$$\rho_3(x_0x_1x_2x_3)=\frac{x_0-x_3}{\rho_2(x_0x_1x_2)-\rho_2(x_1x_2x_3)}+\rho(x_1x_2), \tag{3}$$

and generally a reciprocal difference of *the nth order* for $n+1$ arguments is defined by

$$\rho_n(x_0x_1x_2\ldots x_n) \tag{4}$$

$$=\frac{x_0-x_n}{\rho_{n-1}(x_0x_1\ldots x_{n-1})-\rho_{n-1}(x_1x_2\ldots x_n)}+\rho_{n-2}(x_1x_2\ldots x_{n-1}).$$

Reciprocal differences, like divided differences, are *symmetric functions of the arguments.*

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* **T. N. Thiele**—*Différences réciproques* (*Stzsber. Akad.* **Kopenhagen**), 1906, pp. 153-171; *Interpolationsrechnung*, **Leipzig, 1909, pp. 129-143.**

A table for reciprocal differences can be made in the following manner:

x_0	$f(x_0)$				
		$\rho(x_0x_1)$			
x_1	$f(x_1)$		$\rho_2(x_0x_1x_2)$		
		$\rho(x_1x_2)$		$\rho_3(x_0x_1x_2x_3)$	
x_2	$f(x_2)$		$\rho_2(x_1x_2x_3)$		$\rho_4(x_0x_1x_2x_3x_4)$
		$\rho(x_2x_3)$		$\rho_3(x_1x_2x_3x_4)$	
x_3	$f(x_3)$		$\rho_2(x_2x_3x_4)$		
		$\rho(x_3x_4)$			
x_4	$f(x_4)$				

To obtain the value of a particular entry, say $\rho_3(x_1x_2x_3x_4)$, take the lines sloping upwards and downwards through it. Take the difference x_1-x_4 of the extreme arguments x_1 and x_4, divide it by the difference of the two adjoining entries in the previous column, namely $\rho_2(x_1x_2x_3)-\rho_2(x_2x_3x_4)$, and add to the result the entry which forms a parallelogram with these three entries, namely $\rho(x_2x_3)$. Thus we get

$$\rho_3(x_1x_2x_3x_4)=\frac{x_1-x_4}{\rho_2(x_1x_2x_3)-\rho_2(x_2x_3x_4)}+\rho(x_2x_3).$$

Example. Form the reciprocal differences of $f(x)=x^2$.

x	x^2	ρ	ρ_2	ρ_3
0	0			
		1		
1	1		-2	
		$\frac{1}{3}$		0
2	4		-11	
		$\frac{1}{5}$		0
3	9		-26	
		$\frac{1}{7}$		0.
4	16		-47	
		$\frac{1}{9}$		
5	25			

From the above table we see that reciprocal differences of the third order of x^2 are zero.

7.2. Thiele's Interpolation Formula. Putting x for x_0 in §7.1(1) we have

$$\rho(xx_1)=\frac{x-x_1}{f(x)-f(x_1)},$$

or

$$f(x)=f(x_1)+\frac{x-x_1}{\rho(xx_1)}.$$

Similarly from §7.1(2), 7.1(3),... we have

$$\rho(xx_1)=\rho(x_1x_2)+\frac{x-x_2}{\rho_2(xx_1x_2)-f(x_1)},$$

$$\rho_2(xx_1x_2)=\rho_2(x_1x_2x_3)+\frac{x-x_3}{\rho_3(xx_1x_2x_3)-\rho(x_1x_2)},$$

$$\rho_3(xx_1x_2x_3)=\rho_3(x_1x_2x_3x_4)+\frac{x-x_4}{\rho_4(xx_1x_2x_3x_4)-\rho_2(x_1x_2x_3)},$$

$$\rho_4(xx_1x_2x_3x_4)=\rho_4(x_1x_2x_3x_4x_5)+\frac{x-x_5}{\rho_5(xx_1x_2x_3x_4x_5)-\rho_3(x_1x_2x_3x_4)},$$

and so on.

Hence $f(x)$ is expressed in the form of a continued fraction

$$(1)\quad f(x)=f(x_1)+\frac{x-x_1}{\rho(x_1x_2)+}\,\frac{x-x_2}{\rho_2(x_1x_2x_3)-f(x_1)+}$$

$$\frac{x-x_3}{\rho_3(x_1x_2x_3x_4)-\rho(x_1x_2)+}\,\frac{x-x_4}{\rho_4(x_1x_2x_3x_4x_5)-\rho_2(x_1x_2x_3)+\ldots}$$

$$\frac{x-x_{n-1}}{\rho_{n-1}(x_1x_2\ldots x_n)-\rho_{n-3}(x_1x_2\ldots x_{n-2})+}\,\frac{x-x_n}{\rho_n(xx_1\ldots x_n)-\rho_{n-2}(x_1x_2\ldots x_{n-1})}$$

Here we have taken n interpolation points. If the numerator of a constituent partial fraction vanish, the continued fraction stops there. Putting $x=x_1,\ x_2,\ldots,\ x_n$ in succession, we get from (1),

$$f(x_1),\ f(x_2),\ldots,\ f(x_n)$$

and, as the last constituent partial fraction does not affect the result, the same can be omitted. Thus by Thiele's Interpolation formula* we get a rational function agreeing in value with a given function at the given points.

If $f(x)$ be a rational function, (1) will terminate and the continued fraction will be identically equal to the given function.

Example. Interpolate the value of $f(2{\cdot}5)$ from the table in p. 183. Take $x_1=2$, and by the application of §7.2(1) we get

$$(2{\cdot}5)^2=2^2+\cfrac{{\cdot}5}{{\cdot}2+\cfrac{-{\cdot}5}{-26-4+\cfrac{-1{\cdot}5}{-{\cdot}2}}}$$

$$=4+\cfrac{{\cdot}5}{{\cdot}2+\cfrac{-{\cdot}5}{-30+7{\cdot}5}} \qquad =4+\frac{{\cdot}5}{{\cdot}2+{\cdot}02}$$

$$=4+2{\cdot}25=6{\cdot}25$$

agreeing with the true value.

7.3. Confluent Case : Reciprocal Derivative.

When any two or more of the arguments are equal we get the confluent Reciprocal difference, the most interesting case being when all the arguments are equal. This latter case is known as *Reciprocal Derivative* and is denoted by

$$(1) \qquad r_nf(x)=\lim_{x_1,\,x_2,\dots,\,x_{n+1}\to x}\rho_n(x_1x_2\dots x_{n+1})$$

$$=\rho_n(xxx\dots x).$$

Putting $n=1$, we have

$$(2) \qquad rf(x)=\lim_{x_1,\,x_2\to x}\frac{x_1-x_2}{f(x_1)-f(x_2)}=\frac{1}{f'(x)},$$

i.e., the reciprocal derivative of the first order is the reciprocal of the ordinary derivative.

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* T. N. Thiele—*Interpolationsrechnung*, Leipzig, 1909, p. 132.

Again $r_2 f(x) = \lim\limits_{x_1,\, x_2,\, x_3 \longrightarrow x} \rho_2(x_1 x_2 x_3)$

$$= \lim_{x_1,\, x_2,\, x_3 \longrightarrow x} \left\{ \frac{x_1 - x_3}{\rho(x_1 x_2) - \rho(x_2 x_3)} + f(x_2) \right\}$$

$$= f(x) + \lim_{x_1,\, x_2,\, x_3 \longrightarrow x} \left\{ \frac{(x_1 - x_2) + (x_2 - x_3)}{\rho(x_1 x_2) - \rho(x_2 x_3)} \right\}$$

$$= f(x) + \lim_{x_1,\, x_2,\, x_3 \longrightarrow x} \left\{ \frac{(x_1 - x_2) + (x_2 - x_3)}{\dfrac{1}{f'(x_1)} - \dfrac{1}{f'(x_2)}} \right\}$$

$$= f(x) + 2r \left\{ \frac{1}{f'(x)} \right\}$$

$$= f(x) - \frac{2[f'(x)]^2}{f''(x)}.$$

Similarly

$$r_3 f(x) = \frac{1}{f'(x)} + 3r \left\{ f(x) - \frac{2[f'(x)]^2}{f''(x)} \right\} = \frac{2f'''(x)}{2f'(x)f'''(x) - 3[f''(x)]^2}.$$

In general

$$r_n f(x) = \lim \left\{ \frac{x_1 - x_{n+1}}{\rho_{n-1}(x_1 x_2 \ldots x_n) - \rho_{n-1}(x_2 x_3 \ldots x_{n+1})} + \rho_{n-2}(x_2 x_3 \ldots x_n) \right\}$$

$$= r_{n-2} f(x) + \lim \frac{(x_1 - x_2) + (x_2 - x_3) + \ldots + (x_n - x_{n+1})}{\rho_{n-1}(x_1 x_2 \ldots x_n) - \rho_{n-1}(x_2 x_3 \ldots x_{n+1})}$$

and since $\lim \dfrac{x_1 - x_2}{\rho_{n-1}(x_1 x_2 \ldots x_n) - \rho_{n-1}(x_2 x_3 \ldots x_{n+1})}$

$$= \lim \frac{x_1 - x_2}{r_{n-1} f(x_1) - r_{n-1} f(x_2)} = r . r_{n-1} f(x)$$

we have

$$r_n f(x) = r_{n-2} f(x) + nr\, r_{n-1} f(x), \tag{3}$$

which is the recurrence formula connecting successive reciprocal derivatives.

Putting $f(x) = x^2$, we get

$$r x^2 = \frac{1}{2x}, \quad r_2 x^2 = -3x^2, \quad r_3 x^2 = 0.$$

7.4. Thiele's Theorem. When all the arguments are equal, we get from Thiele's interpolation formula an expression of the function as a continued fraction involving reciprocal derivatives at a point.

The kth divisor in Thiele's interpolation formula is

$$\rho_k(x_1 x_2 \ldots, x_{k+1}) - \rho_{k-2}(x_1 x_2 \ldots {}_{k-1})$$

and in the limit when $x_1, x_2, \ldots, x_{k+1} \longrightarrow x$, we have

$$\lim\{\rho_k(x_1 x_2 \ldots x_{k+1}) - \rho_{k-2}(x_1 x_2 \ldots x_{k-1})\}$$
$$= r_k f(x) - r_{k-2} f(x) = kr\, r_{k-1} f(x), \text{ by §7.3(3).}$$

Hence putting $k=1, 2, 3, \ldots$ in §7.2(1) we have Thiele's theorem,

$$(1) \quad f(x+h) = f(x) + \frac{h}{rf(x)+}\,\frac{h}{2rrf(x)+}\,\frac{h}{3rr_2f(x)+}\,\frac{h}{4rr_3f(x)+\ldots}$$

the last partial divisor being

$$\lim_{x_1, x_2, \ldots, x_n \longrightarrow x} \rho_n(x+h, x_1, x_2, \ldots, x_n) - r_{n-2} f(x).$$

This continued fraction will terminate if the function is rational. Taking $f(x)=x^2$, we have

$$(x+h)^2 = x^2 + \cfrac{h}{\cfrac{1}{2x} + \cfrac{h}{-4x^2 + \cfrac{h}{-\cfrac{1}{2x}}}}.$$

The continued fraction corresponding to $f(x)$ can be put as

$$(2) \quad f(x) = f(0) + \frac{x}{rf(0)+}\,\frac{x}{2rrf(0)+}\,\frac{x}{3rr_2f(0)+}\,\frac{x}{4rr_3f(0)+\ldots}$$

Example. Express e^x as a continued fraction.

We have $\quad re^x = e^{-x}, \qquad r_2e^x = -e^x,$

$r_3e^x = -2e^{-x}, \qquad r_4e^x = e^x,$

$r_5e^x = 3e^{-x}, \qquad r_6e^x = -e^x,$

and it is easily proved by induction that

$$r_{2n}e^x = (-1)^n e^x, \qquad r_{2n+1}e^x = (-1)^n(n+1)e^{-x}.$$

Hence substituting these values in (2) we have

$$e^x = 1 + \cfrac{x}{1 + \cfrac{x}{-2 + \cfrac{x}{-3 + \cfrac{x}{2 + \cfrac{x}{5 + \cfrac{x}{-2 + \cfrac{x}{-7 + \dots}}}}}}}$$

which can be put as

$$e^x = 1 + \frac{x}{1-}\,\frac{x}{2+}\,\frac{x}{3-}\,\frac{x}{2+}\,\frac{x}{5-}\,\frac{x}{2+}\,\frac{x}{7-}\cdots$$

7.5. Aitken's method of Linear Interpolation by Iteration.*

Let $f(x\,;\,a, b, c, d, \dots)$ denote the interpolation polynomial corresponding to the given arguments $a, b, c, d, \dots$ and coinciding in value with $f(x)$ at the points $a, b, c, d, \dots$ Then by Newton's divided difference formula §5.6(4), we have

$$f(x\,;\,a, b) = f(a) + (x-a)f(a, b).$$

Also $$f(x\,;\,a, b) = f(b) + (x-b)f(a, b),$$

since the function is symmetrical with respect to the arguments. Eliminating $f(a, b)$ we get

$$(1) \qquad f(x\,;\,a, b) = \begin{vmatrix} f(a) & a-x \\ f(b) & b-x \end{vmatrix} \div (b-a).$$

This is a first difference approximation to $f(x)$. It is a linear interpolation polynomial.

Again $$f(x\,;\,a, b, c) = f(a) + (x-a)f(a, b) + (x-a)(x-b)f(a, b, c)$$

$$= f(x\,;\,a, b) + (x-a)(x-b)f(a, b, c),$$

and $$f(x\,;\,a, b, c) = f(x\,;\,a, c) + (x-a)(x-c)f(a, b, c).$$

Eliminating $f(a, b, c)$ we get

$$(2) \qquad f(x\,;\,a, b, c) = \begin{vmatrix} f(x\,;\,a, b) & b-x \\ f(x\,;\,a, c) & c-x \end{vmatrix} \div (c-b).$$

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* **A. C. Aitken**,—*Proc. Edin. Math. Soc.* (2), 1932, p. 56.

This is a second difference approximation to $f(x)$ and is a quadratic interpolation polymonial.

Similarly

$$f(x\,;\ a, b, c, d) = f(x\,;\ a, b, c) + (x-a)(x-b)(x-c)f(a, b, c, d)$$

$$= f(x\,;\ a, b, d) + (x-a)(x-b)(x-d)\,f(a, b, c, d)$$

whence

$$(3) \qquad f(x\,;\ a, b, c, d) = \begin{vmatrix} f(x\,;\ a, b, c) & c-x \\ f(x\,;\ a, b, d) & d-x \end{vmatrix} \div (d-c).$$

The above method can be shewn in the following table:

Argument	Function	(1)	(2)	(3)	Parts
a	$f(a)$				$a-x$
b	$f(b)$	$f(x\,;\ a, b)$			$b-x$
c	$f(c)$	$f(x\,;\ a, c)$	$f(x\,;\ a, b, c)$		$c-x$
d	$f(d)$	$f(x\,;\ a, d)$	$f(x\,;\ a, b, d)$	$f(x\,;\ a, b, c, d)$	$d-x$

Each entry in (1) is obtained by cross-multiplication of the corresponding values of the function with the respective parts and divided by the difference of the arguments. Hence it is called a *linear cross-mean*. Each entry in (2) is obtained similarly by cross-multiplication of respective entries in column (1) and parts, and division by the difference of arguments, and so on. Giving to x any definite value we get the approximate value of the interpolation polynomial which coincides with the values of $f(x)$ at the given interpolation points. For the exact value we want the remainder term which has been omitted. Thus if four arguments a, b, c, d are given, we have

$$f(x) = f(x\,;\ a, b, c, d) + (x-a)(x-b)(x-c)(x-d)\,\frac{f^{(4)}(\xi)}{4\,!},$$

where ξ lies in the smallest interval containing a, b, c, d, x.

Sometimes it may be necessary to replace the entries under Parts by numbers proportional to them. The process will terminate when further interpolations do not influence the result.

Example. Find the value of log sin 8° from the following values of log sin θ.

						Parts
6°	9·01923					−2
7°	9·08589	9·15255				−1
9°	9·19433	9·13596	9·144255			+1
10°	9·23967	9·12945	9·14485	9·14366		+2
11°	9·28060	9·123778	9·145357	9·143704	9·14357	+3.

The entries in the columns are calculated thus:

$$\begin{vmatrix} 9{\cdot}01923 & -2 \\ 9{\cdot}08589 & -1 \end{vmatrix} \div 1 = 9{\cdot}15255, \quad \begin{vmatrix} 9{\cdot}01923 & -2 \\ 9{\cdot}19433 & +1 \end{vmatrix} \div 3 = 9{\cdot}13596,$$

and so on.

The successive approximations to log sin 8° are

9·15255, 9·144255, 9·14366 and 9·14357.

Thus log sin 8° = 9·14357,
the true value to five places of decimals being 9·14356.

7.6. Aitken's Quadratic Process. Suppose there is an even number of symmetrically placed arguments

$$a, \quad b, \quad c, \ldots, \quad k,$$
$$-a, \; -b, \; -c, \ldots, \; -k;$$

then

$$f(x\,;\, a,\, -a) = \frac{(a+x)f(a) + (a-x)f(-a)}{2a}. \tag{1}$$

This expression remains unaltered when $-a$ is substituted for a and hence it is an even polynomial in a. We denote it by $f(x\,;\, a^2)$. Thus

$$f(x\,;\, a^2) = \begin{vmatrix} f(a) & x-a \\ f(-a) & x+a \end{vmatrix} \div 2a. \tag{2}$$

Putting $a = x$, we have $f(x\,;\, x^2) = f(x)$.

Now we can proceed with b^2, c^2, d^2,... as in the process of linear interpolation of §7.5. Thus

$$(3) \qquad f(x\,;\,a^2,\,b^2) = \begin{vmatrix} f(x\,;\,a^2) & a^2-x^2 \\ f(x\,;\,b^2) & b^2-x^2 \end{vmatrix} \div (b^2-a^2),$$

$$(4) \qquad f(x\,;\,a^2,\,b^2,\,c^2) = \begin{vmatrix} f(x\,;\,a^2,\,b^2) & b^2-x^2 \\ f(x\,;\,a^2,\,c^2) & c^2-x^2 \end{vmatrix} \div (c^2-b^2),$$

and so on.

Thus we get the following table :

$$\begin{array}{lllll} f(x\,;\,a^2) & & & & a^2-x^2 \\ f(x\,;\,b^2) & f(x\,;\,a^2,\,b^2) & & & b^2-x^2 \\ f(x\,;\,c^2) & f(x\,;\,a^2,\,c^2) & f(x\,;\,a^2,\,b^2,\,c^2) & & c^2-x^2 \\ f(x\,;\,d^2) & f(x\,;\,a^2,\,d^2) & f(x\,;\,a^2,\,b^2,\,d^2) & f(x\,;\,a^2,\,b^2,\,c^2,\,d^2) & d^2-x^2 \end{array}$$

If there are $2n$ arguments, the remainder term will be

$$R_{2n}(x) = (x^2-a^2)(x^2-b^2)\ldots(x^2-k^2)\frac{f^{(2n)}(\xi)}{(2n)\,!}.$$

This process improves the convergence to a marked degree.

When the arguments are equidistant, these will be symmetrical about a middle argument or central origin, m say, and hence subtracting it from each argument and dividing by the interval ω, we get the *prepared* arguments in the form $\pm\cdot5$, $\pm1\cdot5$, $\pm2\cdot5$,... and x is replaced by $\dfrac{x-m}{\omega}$.

Aitken's methods possess particular advantages. We can stop at any point and thus any required degree of approximation can be obtained. When the intervals are unequal, the linear process has considerable practical advantage. For equal interval the quadratic process is to be preferred. Both these methods are very useful for inverse interpolation.

Example. From the following values of $f(x)$, find the value of $f(51)$.

41	·10759	53	·26611
43	·12201	55	·32251
45	·13963	57	·39671
47	·16135	59	·49608

Transferring the origin to 50, we have $a=3$, $b=5$, $c=7$, $d=9$ and $x=1$.

The first column is formed by §7.6(2). Thus

$$f(x\,;\,a^2) = \begin{vmatrix} \cdot 26611 & -2 \\ \cdot 16135 & 4 \end{vmatrix} \div 6 = \cdot 23119$$

$$f(x\,;\,b^2) = \begin{vmatrix} \cdot 32251 & -4 \\ \cdot 13963 & 6 \end{vmatrix} \div 10 = \cdot 249358, \text{ and so on.}$$

The second column is formed by §7.6(3). Thus

$$f(x\,;\,a^2,\,b^2) = \begin{vmatrix} \cdot 23119 & 8 \\ \cdot 249358 & 24 \end{vmatrix} \div 16 = \cdot 222106, \text{ and so on.}$$

Thus we get the table:

·23119				8
·249358	·222106			24
·27898	·221632	·22258		48
·323418	·220942	·22260	·22255	80

Thus $f(x\,;\,a^2,\ b^2,\ c^2,\ d^2) = \cdot 22255$, agreeing with the true value correct to five places of decimals.

7.7. Neville's method of Iteration. A slightly different method has been given by Prof. Neville. He brings the 'Parts' in the second column of the table. The advantage of this method is that repeated arguments are considered in this scheme and hence the derivatives of the function whose values are known are also taken into account.

The table for unequal arguments is represented thus:

Argument	Part				
a	$x-a$	$f(a)$			
			$f(x\,;\,a,\,b)$		
b	$x-b$	$f(b)$		$f(x\,;\,a,\,b,\,c)$	
			$f(x\,;\,b,\,c)$		$f(x\,;\,a,\,b,\,c,\,d)$
c	$x-c$	$f(c)$		$f(x\,;\,b,\,c,\,d)$	
			$f(x\,;\,c,\,d)$		
d	$x-d$	$f(d)$			

To find the value of any entry in the table, we take the lines sloping upwards and downwards from that entry and terminated at the parts. Thus

$$(1) \qquad f(x\,;\,a,b,c)=\begin{vmatrix} x-a & f(x\,;\,a,b) \\ x-c & f(x\,;\,b,c) \end{vmatrix} \div (c-a),$$

$$(2) \qquad f(x\,;\,a,b,c,d)=\begin{vmatrix} x-a & f(x\,;\,a,b,c) \\ x-d & f(x\,;\,b,c,d) \end{vmatrix} \div (d-a),$$

and so on.

For the confluent case, let $a=b$. Then by §5.14(1) we have

$$f(a,a)=f'(a),$$

and we write

$$(3) \qquad f(x\,;\,a,a)=f(a)+(x-a)f'(a)=f(x\,;\,a^2).$$

Similarly if three arguments are equal to a, we write

$$(4) \qquad \begin{aligned} f(x\,;\,a,a,a)&=f(x\,;\,a^2)+(x-a)^2 f(a,a,a) \\ &=f(x\,;\,a^2)+\tfrac{1}{2}(x-a)^2 f''(a)=f(x\,;\,a^3). \end{aligned}$$

The table is represented thus:

$$\begin{array}{llllllll}
a & x-a & f(a) & & & & & \\
 & & & f(x;a^2) & & & & \\
a & x-a & f(a) & & f(x;a^3) & & & \\
 & & & f(x;a^2) & & f(x;a^3,b) & & \\
a & x-a & f(a) & & f(x;a^2,b) & & f(x;a^3,b^2) & \\
 & & & f(x;a,b) & & f(x;a^2,b^2) & & f(x;a^3,b^2,c) \\
b & x-b & f(b) & & f(x;a,b^2) & & f(x;a^2,b^2,c) & \\
 & & & f(x;b^2) & & f(x;a,b^2,c) & & \\
b & x-b & f(b) & & f(x;b^2,c) & & & \\
 & & & f(x;b,c) & & & & \\
c & x-c & f(c) & & & & &
\end{array}$$

The entries are calculated by the relations (1), (2), (3) and (4). Thus

$$f(x\,;\,a^3,b)=\begin{vmatrix} x-a & f(x\,;\,a^3) \\ x-b & f(x\,;\,a^2,b) \end{vmatrix} \div (b-a),$$

$$f(x\,;\,a^3,b^2,c)=\begin{vmatrix} x-a & f(x\,;\,a^3,b^2) \\ x-c & f(x\,;\,a^2,b^2,c) \end{vmatrix} \div (c-a),$$

Example. Find the value of $f(\cdot84)$ from the values given.

·80	·04	·00333			
			·01553		
·81	·03	·00638		·01613	
			·01598		·01613.
·83	·01	·01278		·01613	
			·01623		
·86	$-\cdot02$	·02313			

The entries are calculated thus:

$$\begin{vmatrix} \cdot04 & \cdot00333 \\ \cdot03 & \cdot00638 \end{vmatrix} \div \cdot01 = \cdot01553, \text{ and so on.}$$

The value of $f(\cdot84) = \cdot01613$, which agrees with the true value correct to five places of decimals.

7.8. Inverse Interpolation. The problem of inverse interpolation consists in finding, from a given table of values, the value of the argument corresponding to a given value of the function. In direct interpolation, however, the value of the function is required for some other values of the argument. Thus suppose the values of the function $y=f(x)$ for $x_1, x_2, x_3, \ldots$ are $y_1, y_2, y_3, \ldots$; direct interpolation as we have seen consists in finding the value y_m for $x=x_m$, while inverse interpolation consists in finding the value x_m corresponding to $y=y_m$.

By Newton's formula we have

$$f(x)=f(0)+x\Delta f(0)+\frac{x(x-1)}{2!}\Delta^2 f(0)+\frac{x(x-1)(x-2)}{3!}\Delta^3 f(0)+\ldots$$

and there is only one value of $f(x)$ for a given value of x. But considered as an equation in x, it gives multiple values of x for each given value of $f(x)$. Thus, if $f(x)$ be of the nth degree, we get n values of x. Here we are faced with difficulty of the same nature as in the case of direct interpolation (§4.1). In order to have a result consistent with the given table we assume that within the range of interpolation there should be one value of x corresponding to a given value of y. That is $y=f(x)$ must be a strictly monotone increasing or decreasing function over the range of given values, and that the required value of x must lie within the given range near the centre. Then the inverse function $x=f^{-1}(y)$ will be a single-valued function and can be represented by a polynomial.

7.9. For practical purposes we must have some knowledge of the nature of the graph of $y=f(x)$ and must form a rough estimate of the position of x corresponding to the given value of y, and consider the values in the neighbourhood of this estimate.

Alternatively the columns of arguments and functions can be interchanged, and then the problem of inverse interpolation becomes that of direct interpolation for, unequal intervals, and hence the methods of divided differences can be employed to obtain any intermediate value of x corresponding to a given value of y.

When the given value of $f(x)$ is zero, the problem of inverse interpolation reduces to that of numerical calculation of roots of the equation $f(x)=0$.

7.10. Inverse Interpolation by Successive Approximation.

Writing Gregory-Newton advancing difference formula in the form

$$f(x)-f(0)=x\Delta f(0)+\tfrac{1}{2}x(x-1)\Delta^2 f(0)+\tfrac{1}{6}x(x-1)(x-2)\Delta^3 f(0)+\dots$$

we have

$$(1) \qquad x=\frac{f(x)-f(0)}{\Delta f(0)+\frac{1}{2}(x-1)\Delta^2 f(0)+\frac{1}{6}(x-1)(x-2)\Delta^3 f(0)+\dots}.$$

Since x is small, a first approximation, α_1, is obtained by neglecting terms involving differences of second and higher orders. Thus

$$\alpha_1=\frac{f(x)-f(0)}{\Delta f(0)}.$$

For a second approximation, α_2, substitute the value of α_1 in (1) and neglect differences beyond second order. Thus

$$\alpha_2=\frac{f(x)-f(0)}{\Delta f(0)+\frac{1}{2}(\alpha_1-1)\Delta^2 f(0)}.$$

Similarly a third approximation α_3 is given by

$$\alpha_3=\frac{f(x)-f(0)}{\Delta f(0)+\frac{1}{2}(\alpha_2-1)\Delta^2 f(0)+\frac{1}{6}(\alpha_2-1)(\alpha_2-2)\Delta^3 f(0)},$$

and so on.

Example. Given the present value of an annuity certain for a specified number of years at the following rates of interest :

Rate	2	2·5	3	3·5	4	4·5
Value	16·3514	15·5892	14·8775	14·2124	13·5903	13·0079

Find approximately the rate of interest when the present value will be 14·6061.

The difference table is

2	16·3514				
		−·7622			
2·5	15·5892		·0505		
		−·7117		−·0039	
3	14·8775		·0466		·0003
		−·6651		−·0036	
3·5	14·2124		·0430		·0003
		−·6221		−·0033	
4	13·5903		·0397		
		−·5824			
4·5	13·0079				

Here the interval of differences $\omega={\cdot}5$. Taking the origin at 2, we have

$$14{\cdot}6061=16{\cdot}3514-{\cdot}7622x+\frac{{\cdot}0505x(x-1)}{2!}-\frac{{\cdot}0039x(x-1)(x-2)}{3!}$$
$$+\frac{{\cdot}0003x(x-1)(x-2)(x-3)}{4!}.$$

A first approximation is given by

$$\alpha_1=\frac{14{\cdot}6061-16{\cdot}3514}{-{\cdot}7622}=\frac{-1{\cdot}7453}{-{\cdot}7622}=2{\cdot}2898\ldots$$

For a second approximation, including the next term we have

$$\alpha_2=\frac{-1{\cdot}7453}{-{\cdot}7622+\frac{1}{2}(2{\cdot}2898-1){\cdot}0505}=2{\cdot}3921\ldots$$

Similarly

$$\alpha_3=\frac{-1{\cdot}7453}{-{\cdot}7622+\frac{1}{2}(2{\cdot}3921-1){\cdot}0505-\frac{1}{6}(2{\cdot}3921-1)(2{\cdot}3921-2){\cdot}0089}$$
$$=2{\cdot}3996\ldots$$

The required result is $2+x\omega=2+\frac{1}{2}x$.

Thus the successive approximation to the required result are

$3{\cdot}1449\ldots,\quad 3{\cdot}19605\ldots,\quad 3{\cdot}1998\ldots,$

the true rate of interest being $3{\cdot}2$.

7.11. Inverse Interpolation by Central Differences. It is convenient to use a central difference formula as the basic equation. The coefficients will then be smaller and will converge more quickly. In most cases a first approximation will be more accurate than that obtained from advancing difference formula.

If x lies between $-\frac{1}{4}$ and $\frac{1}{4}$, it is best to use Stirling's formula and if x lies between $\frac{1}{4}$ and $\frac{3}{4}$, Bessel's formula will be helpful.

We have from Stirling's formula

$$u_x-u_0=x\mu\delta u_0+\frac{1}{2}x^2\delta^2u_0+\frac{1}{6}x(x^2-1)\mu\delta^3u_0+\frac{1}{24}x^2(x^2-1)\delta^4u_0+\ldots$$

Dividing throughout by $\mu\delta u_0$ we get

$$(1)\quad x=\frac{u_x-u_0}{\mu\delta u_0}-\frac{1}{2}x^2\frac{\delta^2u_0}{\mu\delta u_0}-\frac{1}{6}x(x^2-1)\frac{\mu\delta^3u_0}{\mu\delta u_0}-\frac{1}{24}x^2(x^2-1)\frac{\delta^4u_0}{\mu\delta u_0}-$$

A first approximation α_1 is obtained by retaining only the first term in (1). Thus

$$\alpha_1=\frac{u_x-u_0}{\mu\delta u_0}.$$

For a second approximation α_2, substitute the value of α_1 in (1). Thus we get

$$\alpha_2=\alpha_1-\frac{1}{2}\alpha_1^2\frac{\delta^2u_0}{\mu\delta u_0}-\frac{1}{6}\alpha_1(\alpha_1^2-1)\frac{\mu\delta^3u_0}{\mu\delta u_0}-\frac{1}{24}\alpha_1^2(\alpha_1^2-1)\frac{\delta^4u_0}{\mu\delta u_0}-\dots$$

A third approximation α_3 is obtained by substituting this value of α_2 in (1) and so on.

Again writing Stirling's formula in the form

$$(2)\quad x=\frac{u_x-u_0}{\mu\delta u_0+\frac{1}{2}x\delta^2u_0+\frac{1}{6}(x^2-1)\mu\delta^3u_0+\frac{1}{24}x(x^2-1)\delta^4u_0+\dots}$$

the successive values can be obtained in the manner of § 7.10.

Thus
$$\alpha_1=\frac{u_x-u_0}{\mu\delta u_0}$$

which is the same in both cases ;

$$\alpha_2=\frac{u_x-u_0}{\mu\delta u_0+\frac{1}{2}\alpha_1\delta^2u_0},\qquad\text{and so on.}$$

Example. Consider the example at p. 196.

Take the origin at 3. The differences to be employed lie on the line

3, 14·8775, − ·6884, ·0466, − ·00375, ·0003

By (1) we have

$$x=\frac{14{\cdot}6061-14{\cdot}8775}{-{\cdot}6884}+\frac{1}{2}\cdot\frac{{\cdot}0466}{{\cdot}6884}x^2-\frac{1}{6}\cdot\frac{{\cdot}00375}{{\cdot}6884}x(x^2-1).$$

Thus
$$\alpha_1=\frac{{\cdot}2714}{{\cdot}6884}={\cdot}3942$$

$$\alpha_2={\cdot}3942+\frac{1}{2}\cdot\frac{{\cdot}0466}{{\cdot}6884}({\cdot}3942)^2-\frac{1}{6}\cdot\frac{{\cdot}00375}{{\cdot}6884}{\cdot}3942\{({\cdot}3942)^2-1\}$$

$$={\cdot}3942+{\cdot}0052+{\cdot}0003$$
$$={\cdot}3997.$$

Thus the successive approximations are

3·1971, 3·19985.

The first value here gives a closer approximation than the second value obtained in p. 197 and the second value than that of the third value. Herein lies the importance of central difference formulae.

Again using the formula (2) we have

$$\alpha_1 = \cdot 3942$$

and
$$\alpha_2 = \frac{-\cdot 2714}{-\cdot 6884 + \frac{1}{2} \times \cdot 3942 \times \cdot 0466} = \cdot 39958.$$

Thus the successive approximations are

$$3{\cdot}1971, \quad 3{\cdot}19979,$$

the true value being 3·2.

7.12. Inverse Interpolation by Divided Differences.

Interchanging the columns of x and y, the problem becomes one of direct interpolation. The table is put thus:

$$\begin{array}{llccc}
\text{Argument} & \text{Function} & & & \\
y_1 & x_1 & & & \\
 & & f(y_1y_2) & & \\
y_2 & x_2 & & f(y_1y_2y_3) & \\
 & & f(y_2y_3) & & f(y_1y_2y_3y_4) \\
y_3 & x_3 & & f(y_2y_3y_4) & \\
 & & f(y_3y_4) & & \\
y_4 & x_4 & & &
\end{array}$$

where
$$f(y_1y_2) = \frac{x_1 - x_2}{y_1 - y_2}, \text{ etc.}$$

By Newton's general formula §5.6(1) we have

$$\begin{aligned}
(1) \qquad x = x_1 &+ (y - y_1)f(y_1y_2) + (y - y_1)(y - y_2)f(y_1y_2y_3) + \dots \\
&+ (y - y_1)(y - y_2)\dots(y - y_{n-1})f(y_1y_2\dots y_n) \\
&+ (y - y_1)(y - y_2)\dots(y - y_n)f(yy_1y_2\dots y_n).
\end{aligned}$$

For numerical calculation of roots of an equation we have

$$(2) \qquad x = x_1 - y_1 f(y_1y_2) + y_1y_2 f(y_1y_2y_3) - \dots$$

Confluent divided differences present a practical method of solving algebraic equations especially when the degree is low. Thus when

$$x_1 = x_2 = x_3 = \dots = x_n,$$

we have by §5.14(1)
$$f(x_1x_2\dots x_n) = \frac{f^{(n-1)}(x_1)}{(n-1)!}$$

so that

$$(3)\qquad f(x)=f(x_1)+(x-x_1)f'(x_1)+\frac{(x-x_1)^2}{2!}f''(x_1)+\dots$$
$$+\frac{(x-x_1)^{n-1}}{(n-1)!}f^{(n-1)}(x_1)+R_n(x).$$

Putting $x_1=0$ in (3) we have

$$(4)\qquad f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\dots,$$

the coefficients being divided differences with repeated argument zero.

Example 1. Following values of $y=f(x)$ are given

x	-42,	-39,	18,	73
y	50,	51,	54,	55.

Find the values of x when $f(x)=53,\ 57.$

Interchanging the columns of x and $f(x)$, the difference table is

y	x			
50	-42			
		3		
51	-39		4	
		19		1
54	18		9	
		55		
55	73			

Hence by (1) we have

$$x=-42+3(y-50)+4(y-50)(y-51)+(y-50)(y-51)(y-54)$$

$\therefore$ For $y=53$, we have $x=-42+3.3+4.3.2+3.2.-1=-15$

and for $y=57$, we have $x=-42+3.7+4.7.6+7.6.3=273.$

Example 2. Find the real root of the equation $x^3+x-3=0$.

Firstly, take four arguments equal to zero. The portion of the table corresponding to these repeated arguments is easily filled up by (4).

The third divided difference is constant, $=1$, and hence the last column is immediately filled up by 1. Other entries are filled up by the relation

$$f(x_1x_2\ldots x_n)=f(x_0x_1\ldots x_{n-1})+(x_n-x_0)f(x_0x_1x_2\ldots x_n).$$

The table stands thus:

0	−3			
		1		
0	−3		0	
		1		1
0	−3		0	
		1		1
0	−3		1	
		2		1
1	−1		2·2	
		4·64		1
1·2	−·072		3·41	
		5·3561		1
1·21	−·018439		3·63	
		5·4287		1
1·22	·035848		3·65	
		5·4652		1
1·22	·035848		3·66	
		5·4652		1
1·22	·035848		3·6534112	
		5·4411284		
1·2134112	−·0000025			

We find the entries corresponding to $x=1$, 1·2, 1·21. These give $f(x)$ negative sign. But $x=1·22$ gives $f(x)$ a positive sign, so the required root lies between 1·21 and 1·22. Repeating the argument 1·22 thrice and calculating the entries, we have

$$·035848+5·4652(x-1·22)+3·66(x-1·22)^2+(x-1·22)^3=f(x)=0.$$

Neglecting the third power of $(x-1·22)$ and solving the quadratic, we have $x=1·2134112$.

Inserting this value in the table and calculating we get

$$f(1·2134112)=-·0000025,$$

so that the root is slightly less than the true value, which is 1·2134116. Thus the value obtained is slightly in error in the last digit, the error being ·0000004.

7.13. Inverse Interpolation by Aitken's method of Iteration.

The methods of interpolation by iteration as explained in § 7.5—7.7 are very useful for inverse interpolation. As the arguments need not be equidistant, we can interchange the columns of argument and function and thence the required result is obtained by the method of iteration.

We give below the example worked out by Aitken.*

Example. Find the positive root of the equation

$$x^7+28x^4-480=0.$$

It is easily seen that the root lies between 1·9 and 2. We have to find the value of x corresponding to $y=0$. Calculating the values of y for $x=1·9$, $1·91$,... the table is formed thus:

y	x			
−25·7140261	1·90			
−14·6254167	1·91	23189586		
− 3·3074639	1·92	2952228	2882864	
+ 8·2439435	1·93	2716929	87312	84138
+20·0329830	1·94	2483678	91702	17 53.

Since $y=0$, the column y represents the 'parts' of the method. The entries in the columns are calculated thus:

$$23189586=\begin{vmatrix} 1·90 & -25·7140261 \\ 1·91 & -14·6254167 \end{vmatrix} \div 11·0886094, \text{ and so on.}$$

Thus we obtain $x=1·922884153$, which is correct to nine places of decimals.

7.14. Interpolation with Two Independent Variables.

A function of two independent variables x and y is usually represented by $f(x\,;\,y)$ or $u_{x,y}$. Let the values of x be x_1, x_2,

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* *Loc. cit*, p. 188. This equation is due to **W. B. Davis**, *Educational Times*, 1867, p. 108. It has been solved by Stirling's formula by **Whittaker and Robinson**, *Calculus of Observations*, p. 61.

$x_3, \ldots$ and those of y be $y_1, y_2, y_3, \ldots$, then the function table is put thus :

x	y_1	y_2	y_3	.	.
x_1	$f(x_1\,;\,y_1)$	$f(x_1\,;\,y_2)$	$f(x_1\,;\,y_3)$	.	.
x_2	$f(x_2\,;\,y_1)$	$f(x_2\,;\,y_2)$	$f(x_2\,;\,y_3)$	.	.
x_3	$f(x_3\,;\,y_1)$	$f(x_3\,;\,y_2)$	$f(x_3\,;\,y_3)$	.	.
.	.	.	.	.	.

From the above table it is seen that it is not necessary for both x and y to vary; x may vary while y remains constant and *vice versa*. Thus interpolations with one particular row or column in the table are interpolations with one variable. The problem consists in obtaining the value of $f(x\,;\,y)$ for particular values of x and y, or in obtaining an approximate expression of $f(x\,;\,y)$ as a function of x and y.

When the arguments proceed by equidistant intervals, say ω and ω', we denote by E_x the operation of increasing the value of x by ω while y remains constant, and by E_y the increment of y by ω' while x remains constant. Thus

$$E_x f(x\,;\,y)=f(x+\omega\,;\,y) \quad \text{and} \quad E_y f(x\,;\,y)=f(x\,;\,y+\omega'),$$

so that $\Delta_x f(x\,;\,y)=f(x+\omega\,;\,y)-f(x\,;\,y)$

and $\quad \Delta_y f(x\,;\,y)=f(x\,;\,y+\omega')-f(x\,;\,y)$.

Again $\Delta_x\Delta_y f(x;y)=f(x+\omega;y+\omega')-f(x+\omega;y)-f(x;y+\omega')+f(x;y)$.

The general formula corresponding to Gregory-Newton formula is

$$(1) \quad f(x+m\omega\,;\,y+n\omega')=E_x^m E_y^n f(x\,;\,y)=(1+\Delta_x)^m(1+\Delta_y)^n f(x\,;\,y)$$

$$=\left\{1+m\Delta_x+\frac{m(m-1)}{2\,!}\Delta_x^2+\ldots\right\}\left\{1+n\Delta_y+\frac{n(n-1)}{2\,!}\Delta_y^2+\ldots\right\}f(x;y)$$

$$=f(x\,;\,y)+(m\Delta_x+n\Delta_y)f(x\,;\,y)$$

$$+\left\{\frac{m(m-1)}{2\,!}\Delta_x^2+mn\Delta_x\Delta_y+\frac{n(n-1)}{2\,!}\Delta_y^2\right\}f(x\,;\,y)$$

$$+\left\{\frac{m(m-1)(m-2)}{3\,!}\Delta_x^3+\frac{m(m-1)}{2\,!}n\Delta_x^2\Delta_y+\frac{m.n(n-1)}{2\,!}\Delta_x\Delta_y^2\right.$$

$$\left.+\frac{n(n-1)(n-2)}{3\,!}\Delta_y^3\right\}f(x\,;\,y)+\ldots$$

This formula is due to Lambert.*

In most cases it will be convenient to transfer the origin to $(0, 0)$ and when the interval of differencing for x and y are the same, to take the interval as unity. The general expression for $f(x\,;\,y)$ can be put as

$$(2)\quad f(x;y)=f(0;0)+(x\Delta_x+y\Delta_y)f(0;0)$$
$$+\left\{\frac{x(x-1)}{2\,!}\Delta_x^2+xy\Delta_x\Delta_y+\frac{y(y-1)}{2\,!}\Delta_y^2\right\}f(0;0)+\ldots$$

Two variable functions are of great frequency in actuarial work. Annuity values of joint lives are generally tabulated for quinquennial values of x and y, and for other values methods of interpolation are to be used.

Example 1. Find an expression for $f(x;y)$ from the following table:

	$y=0$	1	2
$x=0$	7	3	1
1	11	6	
2	17		

Differencing downwards we have $\Delta_x f(0;0)=\ \ 4,\ \Delta_x^2 f(0;0)=2$

,, across $\Delta_y f(0;0)=-4,\ \Delta_y^2 f(0;0)=2.$

Also $\Delta_x\Delta_y f(0\,;\,0)=6-3-11+7=-1$

$$\therefore\quad f(x\,;\,y)=7+4x-4y+\frac{x(x-1)}{2}.2-xy+\frac{y(y-1)}{2}.2$$
$$=x^2-xy+y^2+3x-5y+7.$$

Example 2. Calculate the value of joint-life annuity $f(23\,;\,17)$ from the following table:

	$y=15$	20	25
$x=20$	6·004	4·304	3·325
25	6·029	4·346	
30	6·075		

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* *Beyträge*, Part III. See also **Lagrange**, *Nouv. Mem. de Berlin* (1772) reprinted in *Œuvres*, 3, p. 441.

Here the interval of differencing is 5. Change the origin (20, 15) to (0, 0) and the interval to 1. The required value then is $f(\cdot 6 ; \cdot 4)$

Differencing downwards we have $\Delta_x f = \cdot 025$, $\Delta_x^2 f = \cdot 021$

and ,, across $\Delta_y f = -1\cdot 700$, $\Delta_y^2 f = \cdot 721$.

Also $\Delta_x \Delta_y f(0 ; 0) = 4\cdot 346 - 4\cdot 304 - 6\cdot 029 + 6\cdot 004 = \cdot 017$.

$$\begin{aligned} \therefore\ f(\cdot 6 ; \cdot 4) &= (1 + \cdot 6\Delta_x + \cdot 4\Delta_y - \cdot 12\Delta_x^2 + \cdot 24\Delta_x\Delta_y - \cdot 12\Delta_y^2) f(0 ; 0) \\ &= 6\cdot 004 + \cdot 015 - \cdot 68 - \cdot 00252 + \cdot 00408 - \cdot 08652 \\ &= 5\cdot 25404. \end{aligned}$$

Thus $f(23 ; 17) = 5\cdot 25404$.

7.15. When the two ages concerned are such their sum is a multiple of 5, the origin and the interval may be so chosen that $x + y = 1$.

The general formula then becomes

$$(1)\quad f(x ; 1-x) = f(0 ; 0) + \{x\Delta_x + (1-x)\Delta_y + \tfrac{1}{2}x(x-1)(\Delta_x^2 - 2\Delta_x\Delta_y + \Delta_y^2)\} f(0 ; 0),$$

or

$$(2)\quad f(x ; 1-x) = f(0 ; 1) + x\{f(1 ; 0) - f(0 ; 1)\} + \tfrac{1}{2}x(x-1)\{f(2 ; 0) - 2f(1 ; 1) + f(0 ; 2)\}.$$

Again if $x + y = 2$, the formula becomes

$$f(x; 2-x) = \tfrac{1}{2}x(x-1)f(2; 0) - x(x-2)f(1; 1) + \tfrac{1}{2}(x-1)(x-2)f(0; 2).$$

The problem thus reduces to one of single variable interpolation. We have taken only three terms and it has been found that the three term formula gives a close approximation to the true results as the formulae involving differences of higher order. For a detailed discussion, see Spencer, *J. I. A.*, 40, pp. 293-301.

Example. Consider Example 2 of p. 204.

Changing the origin to (0, 0), (23 ; 17) becomes $(\cdot 6 ; \cdot 4)$ and $x + y = 1$. Hence by (2)

$$\begin{aligned} f(\cdot 6; \cdot 4) &= 4\cdot 304 + \cdot 6(6\cdot 029 - 4\cdot 304) \\ &\quad - \tfrac{1}{2}\cdot 6 \times \cdot 4(6\cdot 075 - 2 \times 4\cdot 346 + 3\cdot 325) \\ &= 4\cdot 304 + 1\cdot 035 - \cdot 08496 = 5\cdot 25404. \end{aligned}$$

7.16. Interpolation with two variables by Divided Differences. When the intervals are not equidistant, interpolation is effected by using divided differences. Divided differences may be formed with respect to x and y separately or simultaneously. A divided difference formed with respect to x with the $r+1$ arguments $x_1, x_2, \ldots x_{r+1}$, and with respect to y with the $s+1$ arguments $y_1, y_2 \ldots y_{s+1}$, is *of order* $r+s$ and is represented by

$$f_{r,s} \equiv f(x_1, x_2 \ldots x_{r+1} \,;\, y_1, y_2, \ldots y_{s+1}).$$

Suppose the values of $f(x;y)$ are tabulated for $x=x_1, x_2, \ldots x_n$ and $y=y_1, y_2, \ldots y_m$.

By Newton's interpolation formula § 5.6(2) we have

$$(1) \quad f(x;y)=f(x_1;y)+\sum_{r=1}^{n-1}(x-x_1)(x-x_2)\ldots(x-x_r)f(x_1,x_2\ldots x_{r+1};y)$$
$$+(x-x_1)(x-x_2)\ldots(x-x_n)f(x,x_1,x_2,\ldots x_n;y)$$

or putting $X_r=(x-x_1)(x-x_2)\ldots(x-x_r)$ and $X_0=1$.

$$(2) \quad f(x;y)=\sum_{r=0}^{n-1}X_r f(x_1, x_2 \ldots x_{r+1};y)+X_n f(x, x_1, x_2 \ldots x_n;y).$$

Again

$$f(x_1, x_2 \ldots x_{r+1};y)=\sum_{s=0}^{m-1}Y_s f(x_1, x_2 \ldots x_{r+1}; y_1, y_2 \ldots y_{s+1})$$
$$+Y_m f(x_1, x_2 \ldots x_{r+1}; y, y_1, y_2 \ldots y_m),$$

where $Y_s=(y-y_1)(y-y_2)\ldots(y-y_s)$ and $Y_0=1$.

Hence

$$(3) \quad f(x;y)=\sum_{r=0}^{n-1}\sum_{s=0}^{m-1}X_r Y_s f_{r,s}+R$$

where

$$(4) \quad R=Y_m\sum_{r=0}^{n-1}X_r f(x_1, x_2 \ldots x_{r+1}; y, y_1, y_2 \ldots y_m)$$
$$+X_n f(x, x_1, x_2 \ldots x_n;y).$$

Again

$$f(x; y, y_1, y_2 \dots y_m) = \sum_{r=0}^{n-1} X_r f(x_1, x_2, \dots x_{r+1}; y, y_1, \dots y_m)$$
$$+ X_n f(x, x_1, x_2 \dots x_n; y, y_1, y_2 \dots y_m)$$

and hence from (4) we have

$$(5) \qquad \begin{aligned} R = X_n f(x, x_1, x_2 \dots x_n; y) + Y_m f(x; y, y_1, y_2 \dots y_m) \\ - X_n Y_m f(x, x_1, x_2 \dots x_n; y, y_1, y_2 \dots y_m). \end{aligned}$$

The remainder term admits of being put in an elegant form. For by §5.7(2) we have

$$f(x, x_1, x_2 \dots x_n; y) = \frac{1}{n!} D_\xi^n f(\xi; y)$$

where ξ lies between the greatest and least of $x, x_1, x_2, \dots, x_n$, and D_ξ denotes partial differentiation. Also

$$f(x, x_1, x_2 \dots x_n; y, y_1, y_2, \dots y_m) = \frac{1}{n!} D_\xi^n f(\xi; y, y_1, y_2, \dots y_m)$$

$$= \frac{1}{n!} D_\xi^n \cdot \frac{1}{m!} D_\eta^m f(\xi; \eta)$$

where η lies between the greatest and least of $y, y_1, y_2 \dots y_m$.

Hence from (5) we get

$$(6) \quad R = \frac{X_n}{n!} D_\xi^n f(\xi; y) + \frac{Y_m}{m!} D_\eta^m f(x; \eta) - \frac{X_n}{n!} \frac{Y_m}{m!} D_\xi^n D_\eta^m f(\xi; \eta),$$

and from (4)

$$(7) \qquad R = \frac{X_n}{n!} D_\xi^n f(\xi; y) + \sum_{r=0}^{n-1} \frac{X_r}{r!} \frac{Y_m}{m!} D_\xi^r D_\eta^m f(\xi; \eta).$$

The values of ξ or η need not be the same in all the terms in R.

The remainder term R vanishes when $x = x_1, x_2, \dots x_n$ and $y = y_1, y_2, \dots y_m$ and (3) will then represent a polynomial of $(n-1)$th degree in x and of $(m-1)$th degree in y.

Writing the first few terms in full we have

$$(8) \qquad \begin{aligned} f(x;y) = f(x_1;y_1) + (x - x_1) f(x_1, x_2; y_1) + (x - x_1)(x - x_2) f(x_1, x_2, x_3; y_1) \\ + (y - y_1) f(x_1; y_1, y_2) + (x - x_1)(y - y_1) f(x_1, x_2; y_1, y_2) \\ + (y - y_1)(y - y_2) f(x_1; y_1, y_2, y_3) \\ + \dots\dots\dots + R. \end{aligned}$$

In a similar manner for three variables x, y, z, we have

$$f(x;y;z)=\sum_{r=0}^{n-1}\sum_{s=0}^{m-1}\sum_{t=0}^{p-1}X_rY_sZ_tf_{r,s,t}+R.$$

7.17. The case for equidistant arguments is obtained from § 7.16 by making the intervals equal. As a particular case, let

$$x_r=r-1,\quad\text{and}\quad y_s=s-1;\quad r=1,2,\ldots n;\ s=1,2,\ldots m.$$

Then by § 5.2(4) we get

$$f_{r,s}=f(0,1,2\ldots.r;0,1,2,\ldots s)=\frac{1}{r!}\Delta_x^rf(0;0,1,2,\ldots s)$$

$$=\frac{1}{r!}\cdot\frac{1}{s!}\Delta_x^r\Delta_y^sf(0;0).$$

Hence from § 7.16(3) we have

$$f(x;y)=\sum_{r=0}^{n-1}\sum_{s=0}^{m-1}\frac{x^{(r)}}{r!}\cdot\frac{y^{(s)}}{s!}\Delta_x^r\Delta_y^sf(0;0)+R, \tag{1}$$

where
$$R=\frac{x^{(n)}}{n!}D_\xi^nf+\frac{y^{(m)}}{m!}D_\eta^mf-\frac{x^{(n)}}{n!}\cdot\frac{y^{(m)}}{m!}D_\xi^nD_\eta^mf.$$

This formula is the same as §7.14(2) with the remainder term.

Again when the arguments of each set are the same, *i.e.*, when $x_1=x_2=\ldots=x_n$ and $y_1=y_2=\ldots=y_m$, we have by § 5.14(1)

$$f(x;y)=\sum_{r=0}^{n-1}\sum_{s=0}^{m-1}\frac{(x-x_1)^r}{r!}\cdot\frac{(y-y_1)^s}{s!}D_{x_1}^rD_{y_1}^sf(x_1;y_1)+R. \tag{2}$$

Further when all the arguments are equal to zero, we have

$$f(x;y)=\sum_{r=0}^{n-1}\sum_{s=0}^{m-1}\frac{x^r}{r!}\cdot\frac{y^s}{s!}D_0^rD_0^sf(0;0)+R. \tag{3}$$

7.18. Another case which is of special interest* is when m depends on r and is connected by the relation

$$m=n-r,\quad r=0,1,2,\ldots n-1.$$

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* O. Biermann—*Vorlesungen über Mathematische Naherungsmethoden*, pp 138-144.

Then

$$f(x;y)=\sum_{r=0}^{n-1}\sum_{s=0}^{n-r-1} X_r Y_s f_{rs} + R. \tag{1}$$

Writing (1) in full we have

$$\begin{aligned} f(x;y) &= f_{00}+(X_1f_{10}+Y_1f_{01})+(X_2f_{20}+X_1Y_1f_{11}+Y_2f_{02}) \\ &\quad +\dots+(X_{n-1}f_{n-1,0}+X_{n-2}Y_1f_{n-2,1}+\dots+Y_{n-1}f_{0,n-1})+R \end{aligned} \tag{2}$$

where by § 7.16(7)

$$R=\frac{X_n}{n!}D_\xi^n f+\frac{X_{n-1}}{(n-1)!}\frac{Y_1}{1!}D_\xi^{n-1}D_\eta f+\dots+\frac{Y_n}{n!}D_\eta^n f. \tag{3}$$

Here we are to calculate fewer terms than in § 7.16(3).

In the limit when the intervals becomes infinitesimal, we get from (2) Taylor's series for two variables.

7.19. Construction of Difference Tables. Taking the function table on p. 203 we first form the divided differences of the values in the first, second, ..., columns of the function table. Thus we get the following *difference table in* x:

$f(x_1;y_1)$	$f(x_1;y_2)$	$f(x_1;y_3)$	...
$f(x_1,x_2;y_1)$	$f(x_1,x_2;y_2)$	$f(x_1,x_2;y_3)$	...
$f(x_1,x_2,x_3;y_1)$	$f(x_1,x_2,x_3;y_2)$	$f(x_1,x_2,x_3;y_3)$	...
.	.	.	.
.	.	.	.

Next we form the divided differences of the first, second, ..., rows of the above table and get the following *difference table in* x *and* y.

$f_{00}=f(x_1;y_1)$	$f_{01}=f(x_1;y_1,y_2)$	$f_{02}=f(x_1;y_1,y_2,y_3)$	..
$f_{10}=f(x_1,x_2;y_1)$	$f_{11}=f(x_1,x_2;y_1,y_2)$	$f_{12}=f(x_1,x_2;y_1,y_2,y_3)$	..
$f_{20}=f(x_1,x_2,x_3;y_1)$	$f_{21}=f(x_1,x_2,x_3;y_1,y_2)$	$f_{22}=f(x_1,x_2,x_3;y_1,y_2,y_3)$	..
.	.	.	.
.	.	.	.

This second table gives the divided differences to be employed in the interpolation formulae.

Example. Find an expression for $f(x; y)$ from the following table:

x	$y=0$	2	3
2	14	70	164
3	24	138	294
5	56	406	746
6	78	630	1104

Forming the difference table in x, we have

2	14	70	164
3	10	68	130
5	2	22	32
6	0	2	3

and thence the difference table in x and y

2	14	28	22
3	10	29	11
5	2	10	0
6	0	1	0

Hence by § 7.16(3) we have

$$\begin{aligned} f(x; y) &= 14 + 10(x-2) + 2(x-2)(x-3) \\ &\quad + y\{28 + 29(x-2) + 10(x-2)(x-3) + (x-2)(x-3)(x-5)\} \\ &\quad + y(y-2)\{22 + 11(x-2)\} \\ &= x^3y + 2x^2 + 11xy^2 - 12xy + 6. \end{aligned}$$

7.20. When the value of $f(x; y)$ is wanted at a given point (x, y) we need not form the second difference table in x and y. From the difference table in x we calculate $f(x; y)$ for the given value of x and then interpolate for the given value of y. Thus to evaluate $f(4; 5)$ from the data of the above example we construct the following table:

y	$f(x; y)$				$f(4; y)$		
0	14	$+10(x-2)$	$+2(x-2)(x-3)$	$+0.(x-2)(x-3)(x-5)$	38		
						104	
2	70	68	22	2	246		44.
						236	
3	164	130	32	3	482		

Hence
$$\begin{aligned} f(4\,;5) &= 38+104y+44y(y-2)\\ &= 38+520+660=1218. \end{aligned}$$

7.21. When the arguments are equidistant, instead of forming divided difference tables, ordinary difference table in x may first be formed and then ordinary difference table in x and y, and the result is obtained by the application of § 7.14(2) or § 7.17(1).

Examples

1. Find the reciprocal differences of x^3, $(1+x^2)^{-1}$, 2^x.

2. Form the reciprocal differences of a^x and shew that

$$\rho_{2n}(x-n,\ x-n+1, \ldots x+n)=(-1)^n a^x,$$

$$\rho_{2n+1}(x-n,\ x-n+1, \ldots x+n+1)=(-1)^n \frac{n+1}{(a-1)a^x}.$$

Hence develop a^x in the form of a continued fraction.

3. Prove that $\rho_2(x_1x_2x_3)=\begin{vmatrix} f(x_2) & f(x_1x_2) \\ f(x_2x_3) & f(x_1x_2x_3)\end{vmatrix} \div f(x_1x_2x_3)$.

4. Form reciprocal differences from the following data:

x	1·568,	1·569,	1·570
$f(x)$	357·61106,	556·69098,	1255·76559,

and hence determine the value of $f(1{\cdot}5685)$.

5. Interpolate the values of $f(7)$ and $f(12)$ by Thiele's formula having given

x	5	10	15	20
$f(x)$	73·35	122·95	170·40	215·85

6. Express $\dfrac{x+2}{3x+4}$ in the form of a continued fraction.

7. Form the reciprocal derivatives of $(1+x)^m$ and shew that

$$r_{2n-1}(1+x)^m=\frac{n(2-m)(3-m)\ldots\ldots(n-m)}{m(1+m)(2+m)\ldots\ldots(n-1+m)}\,(1+x)^{1-m}$$

$$r_{2n}(1+x)^m=\frac{(1+m)(2+m)\ldots(n+m)}{(1-m)(2-m)\ldots(n-m)}(1+x)^m.$$

Hence develop $(1+x)^m$ in the form of a continued fraction.

8. Prove that

$$r_{2n-1} \log x = n^2 x,$$

$$r_{2n} \log x = \log x + 2\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \tfrac{1}{n}\right).$$

Hence deduce that

$$\log(1+x) = \frac{x}{1+}\ \frac{x}{2+}\ \frac{x}{3+}\ \frac{2x}{2+}\ \frac{2x}{5+}\ \frac{3x}{2+}\ \frac{3x}{7+}\ \cdots$$

9. Obtain the reciprocal derivatives of $\tan x$ and shew that

$$r_{4n} \tan x = \tan x$$

$$r_{4n+1} \tan x = (2n+1)(n + \cos^2 x)$$

$$r_{4n+2} \tan x = -\cot x$$

$$r_{4n+3} \tan x = (n+1)(2n+1+2\sin^2 x).$$

Hence deduce that

$$\tan(x+h) = \tan x + \frac{h(\tan x + \cot x)}{\cot x -}\ \frac{h}{1-}\ \frac{h}{3\tan x +}$$

$$\frac{h}{1+}\ \frac{h}{5\cot x -}\ \frac{h}{1-}\ \frac{h}{7\tan x +}\cdots$$

Prove also that

$$\tan x \tan h = 1 - \frac{1}{1+}\ \frac{h}{\cot x -}\ \frac{h}{1-}\ \frac{h}{3\tan x +}$$

$$\frac{h}{1+}\ \frac{h}{5\cot x -}\ \frac{h}{1-}\ \frac{h}{7\tan x +}\cdots$$

10. Determine by Aitken's process the value of $\log \Gamma(\frac{1}{2})$ from the following data :

$\frac{1}{12}$	·74556	$\frac{7}{12}$	·18432
$\frac{3}{12}$	·55938	$\frac{8}{12}$	·13165
$\frac{4}{12}$	·42796	$\frac{9}{12}$	·08828
$\frac{5}{12}$	·32788	$\frac{10}{12}$	·05261

11. Find by iterated method the value of $f(17)$ from the following data :

x	0	5	10	15	20	25	30
$f(x)$	1	1·27628	1·62889	2·07893	2·65330	3·38635	4·32194

12. Find from the following table of annuity values

Age	37	38	39	40
Annuity	13·40	13·16	12·93	12·68

the age corresponding to an annuity of 13·00.

13. The equation $x^3 - 6x - 11 = 0$ has a root between 3 and 4. Obtain it by inverse interpolation correct to three places of decimals.

14. Obtain by inverse interpolation the real root of the equations

(*i*) $x^3 + 2x - 20 = 0,$
(*ii*) $x^3 - 2x - 5 = 0.$

15. From the following values of log x

x	1	2	3	4
log x	0	·30103	·47712	·60206

find x when log $x =$ ·30500.

16. From the following values of $f(x)$

x	11	17	21	23	31
$f(x)$	·14646	·83526	1·94486	2·79846	9·23526

find x when $f(x) = 1·30326$.

17. Calculate the value of joint life annuity $f(31;27)$ from the following table:

	$y=20$	25	30
$x=25$	·532	·555	·586.
30	·469	·489	
35	·413		

18. If $f(0;1)=f(1;0)$ and $f(0;2)=f(2;0)$, shew that

$$f(x;y)=f(0;0)+(x+y)\{\Delta_x+\tfrac{1}{2}(x+y-1)\Delta_x^2\}f(0;0)$$
$$+xy\{f(1;1)-f(2;0)\}.$$

19. Obtain Lagrange's formula of interpolation for two variables

$$f(x;y)=f(\alpha;a)\frac{(x-\beta)(y-b)}{(\alpha-\beta)(a-b)}+f(\alpha;b)\frac{(x-\beta)(y-a)}{(\alpha-\beta)(b-a)}$$
$$+f(\beta;a)\frac{(x-\alpha)(y-b)}{(\beta-\alpha)(a-b)}+f(\beta;b)\frac{(x-\alpha)(y-a)}{(\beta-\alpha)(b-a)}.$$

Hence find the value of $f(31;43)$ from the data:

$f(30;40)=14{\cdot}45$, $f(35;40)=14{\cdot}05$, $f(30;45)=13{\cdot}65$, $f(35;45)=13{\cdot}31$.

20. Find the annual premium $P_{23:32}$ for a joint life insurance from the following data:

	£	s	d		£	s	d		£	s	d
$P_{20:30}=$	2	16	4	$P_{25:30}=$	2	18	10	$P_{30:30}=$	3	2	4
$P_{20:35}=$	3	2	2	$P_{25:35}=$	3	4	2	$P_{30:35}=$	3	7	4
$P_{20:40}=$	3	9	10	$P_{25:40}=$	3	11	6	$P_{30:40}=$	3	14	4

21. Find $f(32;37)$ from the following data:

	$y=30$	35	40
$x=30$	13·930	13·491	12·897
35		13·111	12·581
40			12·125.

22. Given the following annual premiums for endowment assurances for specified term of years:

Age	Term 15 years	20 years	25 years
20	5·947	4·418	3·547
25	6·046	4·530	
30	6·144		

Obtain the annual premium for age 23 for a term of 17 years.

23. Calculate the value of joint life annuity $a_{44:51}$ from the following data:

$a_{40:50}=10{\cdot}894$ $a_{40:55}=9{\cdot}796$ $a_{40:60}=8{\cdot}553$

$a_{45:50}=10{\cdot}591$ $a_{45:55}=9{\cdot}583$

$a_{50:50}=10{\cdot}059$

ANSWERS

CHAPTER I

1. (*i*) $\frac{\sin\theta}{\cos x\theta\cos(x+1)\theta}$. (*ii*) $\frac{-\sin\theta}{\sin x\theta\sin(x+1)\theta}$.

(*iii*) For $\frac{\theta}{2x}$ read $\frac{\theta}{2^x}$; $2^{x+2}\sin\frac{\theta}{2^{x+1}}\left(\sin\frac{\theta}{2^{x+2}}\right)^2$.

(*iv*) $2^{2x+2}\left(\sin\frac{\theta}{2^{x+1}}\right)^4$. (*v*) $-\frac{\tan\frac{\theta}{2^{x+1}}}{\cos\frac{\theta}{2^x}}$. (*vi*) $-\frac{1}{\sin 2^{x+1}\theta}$.

(*vii*) $-2^x\tan\frac{\theta}{2^x}\left(\tan\frac{\theta}{2^{x+1}}\right)^2$. (*viii*) $-\frac{2\left(\sin\frac{\theta}{2^{x+2}}\right)^2}{2^x\sin\frac{\theta}{2^x}}$.

(*ix*) $\frac{\tan\frac{\theta}{2^{x+1}}}{2^{x+1}}$. (*x*) $\frac{1}{2^{2x+1}}-\left\{\frac{\tan\frac{\theta}{2^{x+1}}}{2^{x+1}}\right\}^2$.

(*xi*) $\frac{\sin\frac{\theta}{2^{x+1}}}{\left(2^{x+1}\cos\frac{\theta}{2^{x+1}}\right)^3}$. (*xii*) $\frac{1}{\left(2^{x+1}\cos\frac{\theta}{2^{x+1}}\right)^2}$.

(*xiii*) $-\frac{\log\tan 2^x\theta}{2^{x+1}}$. (*xiv*) $2^x\tan^{-1}\left(\frac{\theta^3}{4.2^{3x}+3.2^x\theta^2}\right)$.

(*xv*) $-\frac{2a^{2^x}}{a^{2^{x+1}}-1}$. (*xvi*) $2^x\,\frac{a^{2^x}-1}{a^{2^x}+1}$.

(*xvii*) $\tan^{-1}\frac{\alpha b-a\beta}{x^2(\alpha^2+a^2)+x(\alpha^2+2\alpha\beta+a^2+2ab)+\alpha\beta+\beta^2+ab+b^2}$.

19. $2(x-2)^n-2(x-3)^n$.

CHAPTER II

1. (*i*) $\frac{(2x-1)(2x+1)(2x+3)(2x+5)(2x+7)}{10}+\frac{21}{2}$.

(*ii*) $\frac{1}{2}-\frac{1}{2(2x+1)}$. (*iii*) $\frac{1}{9}-\frac{3x+2}{3(x+1)(x+2)(x+3)}$.

(*iv*) $\frac{1}{90}-\frac{1}{6(2x+1)(2x+3)(2x+5)}$. (*v*) $\frac{17}{36}-\frac{6x^2+21x+17}{6(x+1)(x+2)(x+3)}$

(*vi*) $\frac{11}{12}-\frac{4x+11}{4(2x+1)(2x+3)}$. (*vii*) $\frac{x(x+1)(x+2)(3x+13)}{12}$.

(*viii*) $\frac{19}{168}-\frac{12x^2+33x+19}{6(3x+1)(3x+4)(3x+7)}$.

(*ix*) $\frac{11}{768}-\frac{2x^2+10x+11}{2(2x+2)(2x+4)(2x+6)(2x+8)}$.

(*x*) $\frac{1}{54}\left\{\frac{39}{10}-\frac{36x+39}{(3x+2)(3x+5)}\right\}$.

(*xi*) Apply § 2.7 (*xii*) Apply § 2.6 (3).

2. (*i*) $2^x(2x^2-4x+6)-6$. (*ii*) $\frac{x(6x^3+16x^2+9x-4)}{3}$.

(*iii*) $\frac{3}{82}\left\{1+\frac{2x-1}{2x+1}8^{2x+1}\right\}$. (*iv*) $\frac{1}{24}-\frac{(-11)^x}{8(2x+1)(2x+3)}$.

3. (*i*) $\frac{1}{20}-\frac{1}{4\{2+(-3)^x\}}$; $\frac{1}{20}$. (*ii*) $1-\frac{1}{(x+1)2^x}$; 1.

(*iii*) $\frac{1}{2}-\frac{1}{(x+1)(x+2)2^x}$; $\frac{1}{2}$. (*iv*) $\frac{x}{\sqrt{2}.\,(x+\sqrt{2})}$; $\frac{1}{\sqrt{2}}$

(*v*) $23-\frac{1}{2^x}(4x^2+16x+23)$; 23.

4. (*i*) $\frac{\pi}{4}-\tan^{-1}\frac{1}{2x+1}$; $\frac{\pi}{4}$,

(*ii*) $\tan^{-1}(34x+13)-\tan^{-1}(13)$; $\tan^{-1}\frac{1}{13}$.

(*iii*) $\tan^{-1}(74x+31)-\tan^{-1}(31)$; $\tan^{-1}\frac{1}{31}$.

(*iv*) $\tan^{-1}(26x+5)-\tan^{-1}5$; $\tan^{-1}\frac{1}{5}$.

5. (*i*) $\frac{8}{3}\left(1-\frac{1}{2^{2x}}\right)+\frac{4}{(\tan 2\theta)^2}-\frac{1}{\left(2^{x-1}\cdot\tan\frac{\theta}{2^{x-1}}\right)^2}$;

$$\frac{8}{3}+\frac{4}{(\tan 2\theta)^2}-\frac{1}{\theta^2}.$$

(*ii*) $\frac{\cos\frac{\theta}{2^{x-1}}}{\left(2^{x-1}\sin\frac{\theta}{2^{x-1}}\right)^3}-\frac{8\cos 2\theta}{(\sin 2\theta)^3}$; $\frac{1}{\theta^3}-\frac{8\cos 2\theta}{(\sin 2\theta)^3}$.

(*iii*) $\frac{1}{4}\left(2^x\sin\frac{\theta}{2^{x-1}}-\sin 2\theta\right)$; $\frac{\theta}{2}-\frac{\sin 2\theta}{4}$.

(*iv*) $\frac{1}{(\sin\theta)^2}-\frac{1}{\left(2^x\sin\frac{\theta}{2^x}\right)^2}$; $\frac{1}{(\sin\theta)^2}-\frac{1}{\theta^2}$.

(*v*) $\left(2^{x-1}\sin\frac{\theta}{2^{x-1}}\right)^2-\left(\frac{\sin 2\theta}{2}\right)^2$; $\theta^2-\left(\frac{\sin 2\theta}{2}\right)^2$.

6. (i) $\frac{2^x\,(a^{2^x}+1)}{a^{2^x}-1}-\frac{a+1}{a-1}$. (ii) $\frac{1}{2}\left\{\frac{a+1}{a-1}-\frac{a^{2^x}+1}{a^{2^x}-1}\right\}$.

7. (i) $\frac{\sin x\theta}{\sin\theta}\cdot\sin x\theta$. (*ii*) $\frac{\sin x\theta}{\sin\theta}\cos x\theta$.

(*iii*) $\cot\frac{\theta}{2}-\cot 2^{x-1}\theta$. (*iv*) $\frac{\tan(x+1)\theta-\tan\theta}{\sin\theta}$.

(v) $\frac{1}{\cos\theta\sin 2(x+1)\theta}\left\{(-1)^{x+1}-\cos 2(x+1)\theta\right\}-\frac{1}{\sin\theta}$.

(vi) $\frac{(-1)^{x+1}+\cos 2(x+1)\theta}{\cos\theta\sin 2(x+1)\theta}+\frac{\tan\theta}{\cos\theta}$.

(*vii*) $\dfrac{1}{2\cos\theta}\left\{\dfrac{1}{\sin\theta.\sin 2\theta}+\dfrac{(-1)^{x+1}}{\sin(x+1)\theta\sin(x+2)\theta}\right\}$.

(*viii*) $\left(\dfrac{2^x}{\sin 2^x\theta}\right)^2-\dfrac{1}{\sin^2\theta}$.

8. (*i*) $8x+\dfrac{14x(x-1)}{2!}+\dfrac{12x(x-1)(x-2)}{3!}+\dfrac{6x(x-1)(x-2)(x-3)}{4!}$.

(*ii*) $1-\dfrac{1}{(x+1)3^x}$. (*iii*) $C-\dfrac{12x^3+36x^2+28x+8}{12x(x+1)(x+2(x+3)}$.

(*iv*) $C-2^{x-2}\dfrac{(x-1)!}{(2x-1)!}$.

9. (*i*) $x^2.2^{x+1}$. (*ii*) $34-\dfrac{4x^2+12x+17}{2^{x-1}}$.

18. $1-3(n+1)!+(n+2)!$

CHAPTER III

2. $C-\dfrac{1}{4x^4}-\dfrac{1}{2x^5}-\dfrac{5}{12x^6}+\dfrac{7}{24x^8}-\dots$

3. $C-\dfrac{1}{16}\left\{\dfrac{1}{x+\frac{1}{4}}+\dfrac{1}{2(x+\frac{1}{4})^2}+\dfrac{1}{6(x+\frac{1}{4})^3}-\dfrac{1}{30(x+\frac{1}{4})^5}\right\}$.
$C=1{\cdot}0787$.

5. $\dfrac{1}{2}\left\{\dfrac{1}{12}+\dfrac{1}{2.12.13}+\dfrac{2}{4.12.13.14}+\dfrac{2.3}{8.12.13.14.15}+\dots\right\}$.

7. $\dfrac{(z+1)^{(n)}-x^{(n)}}{(z-x+1)z^{(n-1)}}$; $\dfrac{z+1}{z-x+1}$.

CHAPTER IV

1. $x^3-9x^2+17x+6$. 2. $-x^3-3x^2-5x+1$. 3. $9-x$.
4. $(x-1)(x-2)(x-3)(x^2+8x+11)$; -47. 5. x^3-2x+1.
6. 2·714, 2·758, 2·801, 2·843, 2·884, 2·924, 2·963, 3·001, 3·037, 3·074, 3·109, 3·143, 3·175.

7. 16·25.

10. Moon's latitude for May 11·0 should read 1°10′10″·6.

12. $f(2)=-74$, $f(4)=-12$. **13.** 9·211526.

14. ·5479.

15. £2 − 10*s* − 8*d*., 2 − 11 − 11, 2 − 13 − 3, 2 − 14 − 8, 2 − 16 − 2, 2 − 17 − 9, 2 − 19 − 5, 3 − 1 − 2, 3 − 2 − 11.

16. 2·871, 2·763, 2·663, 2·570, 2·484, 2·404. **21.** ·9921.

22. 888. **24.** (*i*) 1·1358… (*ii*) 5·34629…

25. (*i*) 0·7853996, (*ii*) 0·5235990. **27.** 0·127451.

28. 46801530. **30.** (*i*) 0·000833319. (*ii*) 0·0049983335.

31. 0·0102512, 0·0034056. **32.** 0·18232156, 0·04879016.

34. 0·0647242. **35.** 36·04. **36.** 594.

38. 977, 961; 1023. **40.** 37·48, 37·49.

CHAPTER V

4. 130326. **5.** $(x-5)^3-2(x-5)^2+(x-5)+1$.

6. $x^5-9x^4+18x^3-x^2+9x-18$ **7.** ·6031444.

9. $f(3)=18$, $f(4)=2548$.

CHAPTER VI

3. 3251. **4.** 3·5282738. **5.** 13·062.

10. ·8; ·00048. **14.** ·01333…; −·0002227…

15. 0·180518. **16.** 320; 4522·5984. **17.** 8·07201.

18. 1. **20.** 1·41808. **21.** 2·82843.

CHAPTER VII

4. 435·47730. **5.** 93·4576, 142·1796. **10.** ·24853.

11. 2·29201. **12.** 38·70. **13.** 3·001…

14. (*i*) 2·4695, (*ii*) 2·09455. **15.** 2·019. **16.** 19.

17. ·487. **19.** 13·90. **20.** £2 − 19*s*. − 3*d*.

21. 13·147. **22.** 5·319. **23.** 10·476.

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